THE FORCE OF MORTALITY FUNCTION*

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1. Introduction. In this paper we shall discuss a function which plays an important role in the theory of life contingencies, especially when limiting processes are involved. The paper will be entirely expository in nature, and no knowledge of actuarial theory will be assumed.

For most of the practical work in the actuarial field, we require a table showing the number living at each integral age n (usually denoted by l_n), the number of deaths in the year of age n to $n+1$ (usually denoted by d_n), and the average death rate over this same year of age (usually denoted by $q_n=d_n/l_n$). Now from a purely mathematical point of view and often for practical reasons as well, we should like to regard these tabulated numbers as the values of continuous functions l_x, d_x, and q_x for integral values n of the variable x. There is nothing about the nature of q_x which would prevent us from assuming that it is a continuous function, but since l_x and d_x denote numbers of persons, they cannot be continuous but, on the contrary, are always discrete. We can, however, assume the existence of a continuous function, l_x, whose values *for integral values of* x are equal to the number living of exact age x and whose intermediate values furnish a close approximation to the number living at such intermediate ages; similarly, we can assume the existence of a continuous function, d_x, whose values *for integral values of* x are equal to the number of deaths in the year of age x to $x+1$, and whose intermediate values furnish a close approximation to the number dying within a year from the attainment of such non-integral ages. We do not have any *a priori* reasons for making these assumptions about the functions l_x, d_x, and q_x, but are forced to rely on the weight of statistical evidence and to point to the well-known philosophical principle, *Natura non agit per saltus.*

When we pass to the continuous picture, we need to have an instantaneous death rate as well as the average death rate q_x. The minute an instantaneous rate is mentioned, one immediately thinks of a derivative, and we shall find that the instantaneous death rate takes the form of a derivative. The number of deaths in a small age interval Δx following age x is $l_x-l_{x+\Delta x}$, and the average death rate during this age interval may be defined as

$$\frac{l_x - l_{x+\Delta x}}{l_x}. \tag{1}$$

But since a year of age is the basic unit of time, we wish to have the average death rate during a year of age following age x. This can be obtained by multiplying (1) by $1/\Delta x$. As usual, we define the instantaneous rate to be the limiting value of the average rate, namely,

$$\lim_{\Delta x \to 0} \frac{l_x - l_{x+\Delta x}}{l_x \cdot \Delta x}. \tag{2}$$

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This instantaneous rate, which is a function of x, is known as the Force of Mortality Function, and in the actuarial literature it is denoted by the symbol μ_x. It is of fundamental importance in actuarial theory as we shall show presently [see also **2**, Chapter 1].

If one compares the limit contained in (2) with the usual definition of a derivative, he finds the numerator turned around and the extra term l_x in the denominator. Thus, we have [see **3**, p. 11].

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{d(\ln l_x)}{dx}. \tag{3}$$

2. Approximate calculation of μ_x. Unless we have a formula expressing l_x in terms of well-known mathematical functions (a situation which will be discussed in §4), it is impossible to determine the form of μ_x from the definition (3). As we have mentioned, all we usually have at our disposal are the values of l_x for integral values of x. Therefore, all we usually can do is approximate the values of μ_x by means of various formulas for numerical differentiation. Thus, if we make the assumption that l_x can be approximated by a polynomial of the second degree (which is a very restrictive assumption), then by Gauss' forward formula [see **1**, p. 64]

$$l_{x+t} = l_x + t\cdot\Delta l_x + \frac{t(t-1)}{2!}\Delta^2 l_{x-1} \tag{4}$$

where the symbol Δ denotes a difference with unit interval. By definition (3), then, we have

$$\mu_x = -\frac{1}{l_x}\left[\frac{dl_{x+t}}{dt}\right]_{t=0} = -\frac{1}{l_x}\left[\Delta l_x - \tfrac{1}{2}\Delta^2 l_{x-1}\right] = \frac{l_{x-1} - l_{x+1}}{2\cdot l_x}. \tag{5}$$

If we make the less restrictive assumption that l_x can be approximated by a polynomial of the fourth degree and again use Gauss' forward formula, then

$$\begin{aligned} l_{x+t} = l_x + t\cdot\Delta l_x + \frac{t(t-1)}{2!}\Delta^2 l_{x-1} + \frac{(t+1)t(t-1)}{3!}\Delta^3 l_{x-1} \\ + \frac{(t+1)t(t-1)(t-2)}{4!}\Delta^4 l_{x-2} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mu_x &= -\frac{1}{l_x}\left[\frac{dl_{x+t}}{dt}\right]_{t=0} = -\frac{1}{l_x}\left[\Delta l_x - \frac{1}{2}\Delta^2 l_{x-1} - \frac{1}{6}\Delta^3 l_{x-1} + \frac{1}{12}\Delta^4 l_{x-2}\right] \\ &= \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12\cdot l_x}. \end{aligned} \tag{7}$$

Formula (7) has been used extensively [see, for example, **3**, p. 397].

Other formulas can be obtained from the relation connecting derivatives and differences, namely, [see **1**, p. 155]

$$\frac{dy_x}{dx} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Delta^k y_x}{k}. \tag{8}$$

If (8) is used in (3), we obtain either

$$\mu_x = \frac{1}{l_x} \sum_{k=1}^{\infty} (-1)^k \frac{\Delta^k l_x}{k} \tag{9}$$

or

$$\mu_x = \sum_{k=1}^{\infty} (-1)^k \frac{\Delta^k (\ln l_x)}{k} \tag{10}$$

in which we may stop at any order of differences indicated by the data at hand [see **3**, p. 14].

3. Use of μ_x in actuarial theory. The function μ_x, as stated above, is of fundamental importance in the field of life contingencies, especially when limiting processes are involved. Let us examine a few illustrations which show the role of this function in actuarial theory.

Consider any age interval x to $x+k$. Following a familiar procedure, let us divide this interval into n subdivisions of length $\Delta t_1, \Delta t_2, \cdots, \Delta t_n$, and let us choose a point in each interval $x+t_1, x+t_2, \cdots, x+t_n$. Since μ_x is the instantaneous death rate at age x, the product

$$\mu_{x+t_j} l_{x+t_j} \Delta t_j \tag{11}$$

is approximately equal to the number of deaths in the jth age interval, an approximation which becomes better as Δt_j is decreased. Then the total number of deaths between ages x and $x+k$ is approximately equal to the sum of all products of the form (11) for $j=1, 2, 3, \cdots, n$; and the total number of deaths is equal to the limiting value of this sum as $\Delta t = \max \{\Delta t_j\}$ shrinks, namely,

$$\lim_{\Delta t \to 0} \sum_{j=1}^{n} \mu_{x+t_j} l_{x+t_j} \Delta t_j = \int_0^k \mu_{x+t} l_{x+t} dt. \tag{12}$$

If in (12) we replace μ_{x+t} by the form (3), we obtain

$$\int_0^k \mu_{x+t} l_{x+t} dt = \int_0^k \left[-\frac{1}{l_{x+t}} \frac{dl_{x+t}}{dt} \right] l_{x+t} dt = -\int_0^k dl_{x+t} = l_x - l_{x+k}, \tag{13}$$

a result which was anticipated since the number of deaths between ages x and $x+k$ is naturally equal to the difference between the number living at age x and those who survive k years later.

When we recall the fact that the integral $\int_a^b f(x)dx$ gives the area under the curve $y=f(x)$ between the ordinates erected at $x=a$ and $x=b$, we see that the

integral (12) gives the area under the curve $y=\mu_x l_x$ between the ordinates erected at $x=x$ and $x=x+k$. Since this area is thus numerically equal to the number of deaths between ages x and $x+k$, the curve whose equation is $y=\mu_x l_x$ is called the Curve of Deaths [see **4**; also see **3**, p. 196].

As a second illustration of the role of the function μ_x in actuarial theory, let us consider the problem of determining a formula for the net single premium for a whole life annuity of \$1 per year payable continuously throughout the remaining lifetime of a person whose present age is x; such a net single premium is usually denoted by $\bar{a}_x$. To prepare for the discussion of this problem, let us first recall the formula for a similar annuity payable annually at the end of each year; such a net single premium is usually denoted by a_x. Assume that l_x persons pay this premium and thus produce a fund of $l_x \cdot a_x$. At the end of t years, since l_{x+t} persons survive, l_{x+t} dollars are paid out. If the fund earns interest at the rate i, compounded annually, the present value of this payment is $v^t l_{x+t}$ where v is the discount factor $(1+i)^{-1}$. The present value of all payments to be made until the last survivor dies is therefore $\sum_{t=1}^{\omega-x-1} v^t l_{x+t}$ where ω is the smallest age for which $l_x=0$. Since the present value of all payments to be made must equal the fund originally set up, we obtain after dividing by l_x

$$a_x = \frac{1}{l_x} \sum_{t=1}^{\omega-x-1} v^t l_{x+t}. \tag{14}$$

Returning to the annuity payable continuously, we note that the fund originally set up in this case is $l_x \cdot \bar{a}_x$. Let us divide the interval from age x to age ω as described in the second paragraph of this section. Since \$1 is paid out to each survivor in a year's time, Δt_j dollars are paid to each survivor in the interval of time Δt_j. Thus, the present value of the total amount paid out in the jth age interval is approximately equal to

$$v^{t_j} l_{x+t_j} \Delta t_j$$

which means that the usual reasoning will lead us to the result

$$l_x \cdot \bar{a}_x = \lim_{\Delta t \to 0} \sum_{j=1}^{n} v^{t_j} l_{x+t_j} \Delta t_j = \int_0^{\omega-x} v^t l_{x+t} dt$$

or

$$\bar{a}_x = \frac{1}{l_x} \int_0^{\omega-x} v^t l_{x+t} dt. \tag{15}$$

Unless l_x can be expressed as a simple function, we must resort to approximate integration formulas in order to evaluate the integral on the right in (15). For example, if the Euler-Maclaurin formula is used as far as the term involving the first derivative [see **1**, pp. 187 ff.], we obtain

$$\bar{a}_x = \frac{1}{l_x} \left\{ \sum_{t=0}^{\omega-x-1} v^t l_{x+t} + \frac{1}{2} (v^{\omega-x} l_\omega - l_x) - \frac{1}{12} \left[\frac{d}{dt} (v^t l_{x+t}) \right]_{t=0}^{t=\omega-x} \right\}.$$

$$(16)\qquad \bar{a}_x = \frac{1}{l_x}\sum_{t=1}^{\omega-x-1} v^t l_{x+t} + \frac{1}{2} - \frac{1}{12\cdot l_x}\left[v^t \frac{dl_{x+t}}{dt} + v^t l_{x+t}\ln v\right]_{t=0}^{t=\omega-x}$$

$$= a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$$

where $\delta = -\ln v$ is the so-called force of interest, the nominal rate which compounded continuously produces the given effective rate i [see 3, p. 133].

We might digress for a moment at this point to note the analogy between the function μ_x and the quantity δ mentioned above. If we have an effective annual interest rate i, the amount of interest earned in a small interval of time Δx is

$$(1+i)^{\Delta x} - 1.$$

If we multiply this by $1/\Delta x$, we obtain the total interest which would be earned in one year under the assumption that like amounts of interest are earned in each interval of length Δx; that is, we obtain the nominal annual rate which if converted $1/\Delta x$ times per year gives the effective rate i. We define the nominal annual rate which if converted continuously gives the effective rate i to be the limiting value of this expression, namely,

$$\lim_{\Delta x\to 0}\frac{(1+i)^{\Delta x}-1}{\Delta x} = \lim_{\Delta x\to 0}(1+i)^{\Delta x}\cdot\ln(1+i) = \ln(1+i) = \delta.$$

If we compare this discussion with that preceding formula (2), we see that δ plays a role in the theory of compound interest similar to that played by the function μ_x in the theory of life contingencies.

As a final illustration of the role of the function μ_x in actuarial theory, let us consider the problem of determining the net single premium for a whole life insurance of \$1 payable at the moment of death of a person whose present age is x; this is usually denoted by $\bar{A}_x$. In this case, the fund originally set up is $l_x\cdot\bar{A}_x$. Let us again divide the interval from age x to age ω as described above. As before, the product (11) gives us approximately the number of deaths in the jth age interval, and since the amount of each insurance is \$1, this product also gives us approximately the total number of dollars paid out in claims during the jth age interval. The present value of this amount is

$$v^{t_j}\mu_{x+t_j}l_{x+t_j}\Delta t_j$$

which means that the usual reasoning will lead us to the result

$$l_x\cdot\bar{A}_x = \lim_{\Delta t\to 0}\sum_{j=1}^{n} v^{t_j}\mu_{x+t_j}l_{x+t_j}\Delta t_j = \int_0^{\omega-x} v^t\mu_{x+t}l_{x+t}dt$$

or

$$(17)\qquad \bar{A}_x = \frac{1}{l_x}\int_0^{\omega-x} v^t\mu_{x+t}l_{x+t}dt.$$

When we use (3) in the integral of (17), we get

$$\overline{A}_x = -\frac{1}{l_x}\int_0^{\omega-x} v^t \frac{dl_{x+t}}{dt}\,dt = -\frac{1}{l_x}\int_0^{\omega-x} v^t dl_{x+t}.$$

The last integral suggests an integration by parts which leads to the result

$$\begin{aligned}\overline{A}_x &= -\frac{1}{l_x}\left[v^t l_{x+t}\right]_{t=0}^{t=\omega-x} + \frac{\ln v}{l_x}\int_0^{\omega-x} v^t l_{x+t}\,dt \\ &= -\frac{1}{l_x}(v^{\omega-x}l_\omega - l_x) - \delta\bar{a}_x \qquad (18) \\ &= 1 - \delta\bar{a}_x \qquad \text{[see 3, p. 152].}\end{aligned}$$

4. The law of mortality. Those who work in the empirical sciences feel the greatest satisfaction when the data they obtain follow the pattern of some familiar mathematical function. In §1 we assumed the existence of a continuous function l_x and from it defined the derived function μ_x. A number of attempts have been made to represent l_x and μ_x by means of familiar mathematical functions, and although a perfect fit has never been obtained, three of these attempts are of particular interest.

The first noteworthy proposal was made in 1825 [7] by Benjamin Gompertz (1779–1865), a British actuary who, we might mention, also made some contributions to astronomy and pure mathematics [5]. He stated his basic assumption as follows: "A person's resistance to death decreases as his age increases in such a way that at the end of equally infinitely small intervals of time he loses equally infinitely small proportions of his remaining power to oppose destruction." Mathematically, this statement is equivalent to the assumption that the force of mortality increases in geometrical progression as x increases, or that

$$\mu_x = Bc^x \qquad (19)$$

where B and c are constants. Formula (19) is known as Gompertz's Law.

In 1860, an important modification of this proposal [8] was made by William Matthew Makeham (18?–1893), another British actuary, who over a period of thirty-five years contributed many brilliant papers to the *Journal of the Institute of Actuaries* [6]. He assumed that the force of mortality consists not only of a part which increases in geometrical progression but also contains a part which remains constant throughout life; that is,

$$\mu_x = A + Bc^x \qquad (20)$$

where A, B, and c are constants. This modification of Gompertz's Law is known as Makeham's Law.

Makeham later proposed a second modification [9] in which the force of mortality was assumed to contain a third part which increases in arithmetical progression as x increases; that is,

$$\mu_x = A + Bc^x + Hx. \qquad (21)$$

Of these three proposals, formula (20) or Makeham's Law is of greatest importance. It fits many important tables at ages 20 and over. Furthermore, if a table follows Makeham's Law, much of the practical work with the table is simplified materially. In fact, the saving in labor is so great that some sets of mortality statistics have been graduated by means of the Makeham formula even though the formula did not fit the original data too closely.

It should be noted at this point that Gompertz's Law is the special case of Makeham's Law in which $A=0$. Therefore, everything we have said and shall say about the latter is also true about the former. If a table follows Gompertz's Law, the simplification mentioned above is even greater. While Makeham's Law does have wider application, an annuity mortality table now used by many insurance companies, the 1937 Standard Annuity Table, follows Gompertz's Law at most ages.

5. The law of uniform seniority. We shall show the basic reason for the great simplification mentioned above. If in equation (20) we substitute (3) for μ_x, we obtain

$$-\frac{d(\ln l_x)}{dx} = A + Bc^x \tag{22}$$

whence

$$\ln l_x = -\int (A + Bc^x)dx = -Ax - \frac{Bc^x}{\ln c} + C \tag{23}$$

where C is the constant of integration. If in (23) we replace $-A$ by $\ln s$, $-B/(\ln c)$ by $\ln g$, and C by $\ln k$, then l_x takes the form

$$l_x = ks^x g^{c^x} \qquad \text{[see 3, p. 192].} \tag{24}$$

In (24), it should be noted, s, g, and c are constants which depend upon the mortality statistics themselves while k merely plays the role of the constant of integration.

Now of the l_x lives of age x, l_{x+t} will survive to age $x+t$. Thus, under Makeham's assumption, the probability that a life of age x will survive t years (which is usually denoted by ${}_tp_x$) is

$${}_tp_x = \frac{l_{x+t}}{l_x} = \frac{ks^{x+t}g^{c^{x+t}}}{ks^x g^{c^x}} = s^t g^{c^x(c^t-1)}. \tag{25}$$

Likewise the probability that a life of age y will survive t years is

$${}_tp_y = s^t g^{c^y(c^t-1)}. \tag{26}$$

If we assume that the survival of the one life is independent of the survival of the other, the probability that both will survive t years (which is usually denoted by ${}_tp_{xy}$) is the product of (25) and (26), namely,

$${}_tp_{xy} = s^{2t} g^{(c^x+c^y)(c^t-1)}. \tag{27}$$

Now, it is possible to determine an age w such that

$$2c^w = c^x + c^y. \tag{28}$$

In fact, if we let x denote the younger of the two ages, factor c^x from the right side of (28), and then take the logarithm of both sides. We obtain for w

$$w = x + \frac{\ln (1 + c^{y-x}) - \ln 2}{\ln c}. \tag{29}$$

Our result (29) shows not only that it is possible to obtain the desired age w but also that the difference $w-x$ is a function only of the difference $y-x$ and does not depend upon the individual values of x and y themselves. This fact is called the Law of Uniform Seniority [see 3, p. 258]. Thus, in order to determine w easily in practice, all we need is a table of the values of the difference $w-x$ corresponding to the commonly-occurring values of the difference $y-x$.

With w determined, we have upon using (28) in (27)

$${}_tp_{xy} = s^{2t}g^{2c^w(c^t-1)} = s^{2t}g^{(c^w+c^w)(c^t-1)} = {}_tp_{ww} \tag{30}$$

which shows that we need to calculate the values of this function of two age variables only for equal values of the variables.

The same type of result is obtained when one studies the other actuarial functions of two age variables. The results, moreover, are not confined to functions of only two age variables. One can make the general statement that whenever Makeham's Law holds, any problem involving several lives can be reduced to one in which the ages of those lives are all equal. Thus, if a table contains 100 ages and we have a problem involving n lives, we need to prepare a table of 100 values of each actuarial function required instead of a table of 100^n values which would have to be calculated if all combinations of ages had to be used. This fact illustrates very forcibly the great saving in labor which results whenever Makeham's Law holds.

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SOME CONVEXITY PROPERTIES OF SURFACES OF NEGATIVE CURVATURE

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1. Introduction. A very simple formula occurring in the first semester of a course in differential geometry is the expression (2), below, for the Gaussian, or total, curvature K of a surface S. The purpose of the present note is the analysis, by means of elementary calculus, of the implications of (2) in the two cases $K \leqq 0$ and $K \geqq 0$.

In the neighborhood of a point where $K < 0$, the surface lies on both sides of the tangent plane, as in the depressions between one's knuckles; accordingly, a surface S with $K \leqq 0$ at each of its points is called a *saddle surface*. In the neighborhood of a point where $K > 0$, the surface lies entirely on one side of the tangent plane, as on the tops of one's knuckles. A visualization of the surfaces involved should make our result intuitively more plausible.

2. Geodesic parameters. If an analytic surface S is given in parametric representation,

$$S: \quad x_j = x_j(u, v), \qquad (j = 1, 2, 3),$$

(u, v) in a domain D, then the element of length ds and the element of area da are given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

and

$$da = (EG - F^2)^{1/2}dudv,$$

respectively, where E, F, and G are the coefficients of the first fundamental differential form for S:

$$E = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \qquad F = \sum_{j=1}^{3}\frac{\partial x_j}{\partial u}\frac{\partial x_j}{\partial v}, \qquad G = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2.$$

Given a smooth curve C_0 on S, there exists a unique family of geodesics on S intersecting C_0 orthogonally; if segments of equal length s be measured along the geodesics from C_0, then the locus of their end-points is an orthogonal trajectory C_s of the geodesics. Parameters u, v, called *geodesic parameters*, can then be chosen such that the coefficients of the first fundamental form satisfy

$$E = 1, \qquad F = 0, \qquad G = [\mu(u, v)]^2, \qquad (\mu \geqq 0), \tag{1}$$

so that we have

$$ds^2 = du^2 + \mu^2 dv^2$$

and

$$da = \mu dudv.$$

The curves $v=$a constant$=v_0$ are the geodesics, of length u_2-u_1, between the points (u_1, v_0) and (u_2, v_0) on S, while the curves $u=$a constant are the *geodesic parallels* C_v [2, p. 206]. The surface is said to be given in *geodesic representation.*

The geodesic representation holds for the part S^* of S which is physically covered by the family of geodesics. It is to such a part S^*, which we shall henceforth denote S, that our results apply. The mapping is one-to-one between the points of the (u, v)-domain of definition and the points of S insofar as the geodesics of the family do not intersect; but we do not necessarily restrict our attention to such a part of the surface.

Singular points of the surface, or of the geodesic family, are points where $\mu=0$; the remaining points, where $\mu>0$, are *regular points.*

The Gaussian curvature K of S exists at all regular points of S.

By a classical theorem of Gauss [2, p. 155], K is a function of E, F, and G, and their partial derivatives of the first and second orders. If (1) is satisfied, the formula reduces [3] to

$$K = -\frac{1}{\mu}\frac{\partial^2\mu}{\partial u^2}. \tag{2}$$

3. Convex functions. A function $f(u)$, defined in an interval, is said to be *convex* there provided we have

$$f[ta + (1-t)b] \leqq tf(a) + (1-t)f(b)$$

for all a, b in the interval and for all t satisfying $0<t<1$; that is, provided each chord of the curve lies nowhere below the curve.

It follows that if $f(u)$ is convex in an open interval, then $f(u)$ is continous there.

If a function $f(u)$ is continuous in an interval, then $f(u)$ is convex there if any and only if, for all u and h such that $u+h$ and $u-h$ are in the interval, we have

$$f(u) \leqq \tfrac{1}{2}[f(u+h) + f(u-h)]. \tag{3}$$

If $f''(u)$ exists throughout an interval, then $f(u)$ is convex there if and only if we have $f''(u)\geqq 0$ at each point of the interval.

4. Lemma. We shall say that S is a *surface of non-positive Gaussian curvature* provided we have $K\leqq 0$ at all regular points of S.

LEMMA. *If an analytic surface S is given in geodesic representation, then a necessary and sufficient condition that S be a surface of non-positive Gaussian curvature is that the function $\mu(u, v_0)$ be a convex function of u for each line-segment $u_1<u<u_2$, $v=v_0$, in the (u, v)-domain of definition D.*

Proof. The function $\mu(u, v)$ has continuous second partial derivatives at all regular points of S.

If $\mu(u, v_0)$ is a convex function of u, then we have $\partial^2\mu/\partial n^2\geqq 0$ at all regular

points; therefore, by (2), we have $K \leqq 0$ at these points.

Conversely, if $K \leqq 0$ at all regular points of S, then, by (2) μ satisfies the convexity property $\partial^2\mu/\partial u^2 \geqq 0$ at these points. Further, since the non-negative function μ satisfies $\mu = 0$ at all singular points, it follows that for any singular point (u_0, v_0), μ satisfies the convexity inequality

$$\mu(u_0, v_0) \leqq \tfrac{1}{2}[\mu(u_0 + h, v_0) + \mu(u_0 - h, v_0)].$$

Therefore, by (3), since μ is continuous, $\mu(u, v_0)$ is a convex function of u.

We note parenthetically that if S is given in isothermic representation,

$$E = G = \lambda(u, v), \qquad F = 0,$$

instead of in geodesic representation, then a somewhat analogous situation holds. For now K is given by

$$K = \frac{-1}{2\lambda}\left(\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2}\right),$$

so that S is a surface of non-positive Gaussian curvature if and only if $\log \lambda$ satisfies

$$\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2} \geqq 0$$

at all regular points; that is, if and only if $\log \lambda$ is a subharmonic function of (u, v) [1, 4]. Subharmonic functions furnish a natural generalization of convex functions to functions of more than one variable. A consideration of subharmonic functions leads to a number of results concerning surfaces of non-positive Gaussian curvature; for example, it can be shown by this means that the isoperimetric inequality

$$a \leqq \frac{1}{4\pi} l^2 \tag{4}$$

characterizes surfaces of non-positive Gaussian curvature.

5. On geodesic parallels. We shall prove the following theorem.

THEOREM 1. *Let the arcs $C(u)$, $u_0 \leqq u \leqq u_1$, of length $l(u)$, be arcs of geodesic parallels between geodesics $v = v_0$ and $v = v_1$, $v_0 < v_1$, on a surface S of non-positive Gaussian curvature. Then the length $l(u)$ is a convex function of u (that is, of the geodesic length $u - u_0$); $l(u)$ is strictly convex if S is not a developable surface, and is linear if S is developable.*

Proof. By Lemma 1, $\mu(u, c)$ is a convex function of u, for $v_0 < c < v_1$. Hence for a, b, c, and t satisfying $u_0 < a < b < u_1$, $v_0 < c < v_1$, and $0 < t < 1$, we have

$$\mu[(ta + (1 - t)b), c] \leqq t\mu(a, c) + (1 - t)\mu(b, c),$$

the sign of equality holding if and only if $\partial^2\mu/\partial u^2 \equiv 0$ on the line-segment $u_0 < u < u_1$, $v = c$; that is, by (2), if and only if $K \equiv 0$ there. Hence

$$l(ta + (1-t)b) = \int_{v_0}^{v_1} \mu[(ta + (1-t)b), v]dv$$

$$\leqq \int_{v_0}^{v_1} [t\mu(a, v) + (1-t)\mu(b, v)]dv = tl(a) + (1-t)l(b),$$

the sign of equality holding if and only if $K \equiv 0$ for $u_0 < u < u_1$, $v_0 < v < v_1$, and therefore, by analytic continuation, $K \equiv 0$ on S; that is, if and only if S is a developable surface.

COROLLARY. *If an arc C of length l on an analytic surface S of non-positive Gaussian curvature is translated geodesically arbitrary distances in the two directions on S from C, then at least one of the resulting arcs is of length not less than l; the function $l(u)$ can have an interior maximum only if $l(u)$ is identically constant.*

We note, regarding the corollary, that if $l(u)$ is identically constant then S must be developable; but the hypothesis that S is developable does not imply that $l(u)$ is constant.

THEOREM 2. *Let the arcs $C(u)$, $u_0 - W < u < u_0 + W$, of length $l(u)$, be arcs of geodesic parallels between geodesics $v = v_0$ and $v = v_1$, $v_0 < v_1$, on an analytic surface S of non-positive Gaussian curvature, and let $a(w)$ denote the area of the part of S enclosed by $v = v_0$, $C(u_0 + w)$, $v = v_1$, and $C(u_0 - w)$, $0 \leqq w < W$. Then $a(w)$ is a convex function of w; $a(w)$ is strictly convex if S is not a developable surface, and is linear if S is developable.*

Proof. We have

$$a(w) = \int_{u_0-w}^{u_0+w} \int_{v_0}^{v_1} \mu(u, v)dudv = \int_{u_0-w}^{u_0+w} l(u)du,$$

whence

$$a'(w) = l(u_0 + w) + l(u_0 - w),$$
$$a''(w) = l'(u_0 + w) - l'(u_0 - w).$$

Since, by Theorem 1, $l(u)$ is convex, it follows that $l'(u)$ is monotonic non-decreasing, so that we have $a''(w) \geqq 0$, the sign of equality holding if and only if $l'(u) \equiv$ a constant; that is, since

$$l''(u) = \int_{v_0}^{v_1} \frac{\partial^2\mu}{\partial u^2} dv,$$

if and only if S is developable.

6. Geodesic polar coördinates. If equal lengths are laid off from a point

P_0 of S along the geodesics in all directions through P_0 on S, then the locus of end-points is an orthogonal trajectory of the geodesics. We take the geodesics for the curves $v=$a constant, and let u denote distances measured along the geodesics from P_0. The coördinate v can be adjusted to satisfy $v_1=\theta_1$, for all θ_1, where θ_1 is the angle between the tangents at P_0 to the geodesics $v=0$ and $v=v_1$.

We shall write r, θ for u, v, respectively.

Necessary and sufficient conditions for this coördinate system are (1) and

$$\mu(0, \theta) = 0, \qquad \left[\frac{\partial \mu}{\partial r}\right]_{r=0} = 1. \tag{5}$$

Now r, θ are called *geodesic polar coördinates*, with *pole* at P_0; the curve $r=r_0$ is a *geodesic circle* with *center* at P_0 and *geodesic radius* r_0 [2, p. 208].

7. Definitions. Several functions of geometrical significance involving the geodesic radius have certain properties in common. It is convenient to collect these properties in the following definitions.

CONDITION A. For a given surface S of non-positive Gaussian curvature, and for a given pole P_0 of geodesic polar coordinates on S, a function $\phi(r)$ of the geodesic radius r satisfies Condition A provided: $\phi(0)=0$; for $r\geqq 0$ on S, $\phi(r)$ is a continuous monotonic non-decreasing convex function of r; $\phi(r)\equiv 0$ if S is a developable surface, but otherwise $\phi(r)$ is monotonic increasing and strictly convex.

CONDITION B. For a given surface S of non-positive Gaussian curvature, and for a given pole P_0 of geodesic polar coordinates on S, a function $\psi(r)$ of the geodesic radius r satisfies Condition B provided $\psi(0)=1$; for $r\geqq 0$ on S, $\psi(r)$ is a continuous monotonic non-decreasing function of r; $\psi(r)\equiv 1$ if S is a developable surface, but otherwise $\psi(r)$ is monotonic increasing.

Thus if $\phi(r)$ satisfies Condition A, then $\phi(r)+1$ satisfies Condition B; but $\psi(r)$ might satisfy Condition B without $\phi(r)-1$ satisfying Condition A.

We note that if a function $\phi(r)$ satisfies Condition A, then $[\phi(r)]^\alpha$, $\alpha>1$, also satisfies Condition A. If $\psi(r)$ satisfies Condition B, then $[\psi(r)]^\beta$, $\beta>0$, also satisfies Condition B.

8. On geodesic circles. We shall establish the following result concerning the length

$$l(r) \equiv \int_0^{2\pi} \mu(r, \theta)\, d\theta$$

of the circumference of geodesic circles on surfaces of non-positive Gaussian curvature.

THEOREM 3. *Let S be an analytic surface of non-positive Gaussian curvature, and let $l(r)$ denote the length of the circumference of the geodesic circle on S with fixed*

center P_0 and geodesic radius r. Then the function

$$\phi_1(r) \equiv l(r) - 2\pi r, \qquad (r \geqq 0 \text{ on } S),$$

satisfies Condition A, and the function

$$\psi_1(r) \equiv \frac{l(r)}{2\pi r}, \qquad (r > 0 \text{ on } S),$$

$$\psi_1(0) \equiv 1$$

satisfies Condition B.

Proof. Since, by Lemma 1, for each θ_0, $0 \leqq \theta_0 < 2\pi$, $\mu(r, \theta_0)$ is a convex function of r and satisfies (5), we have

$$\left[\frac{\partial \mu}{\partial r}\right]_{r>0} \geqq \frac{\mu}{r} \geqq 1, \tag{6}$$

the signs of equality holding if and only if $\partial\mu/\partial r \equiv 1$ on the geodesic $\theta = \theta_0$; that is, by (2), if and only if $K \equiv 0$ on $\theta = \theta_0$.

From (6) it follows that the function

$$\phi_0(r, \theta_0) \equiv \mu(r, \theta_0) - r, \qquad (r \geqq 0 \text{ on } S),$$

which vanishes at $r = 0$, satisfies $\partial\phi_0/\partial r \geqq 0$, so that $\phi_0(r, \theta_0)$ is a continuous monotonic non-decreasing convex function of r. And $\phi_0(r, \theta_0)$ is monotonic increasing and strictly convex unless $K \equiv 0$ on $\theta = \theta_0$.

It follows (see the proof of Theorem 1) that the function

$$\phi_1(r) \equiv l(r) - 2\pi r = \int_0^{2\pi} \phi_0(r, \theta)\, d\theta$$

satisfies Condition A.

By (5) and l'Hospital's rule, the function

$$\psi_0(r, \theta_0) \equiv \frac{\mu(r, \theta_0)}{r}, \qquad (r > 0 \text{ on } S),$$

$$\psi_0(0, \theta_0) \equiv 1$$

is continuous at $r = 0$. Differentiating $\psi_0(r, \theta_0)$ with respect to r, we see by (6) that $\psi_0(r, \theta_0)$ is monotonic non-decreasing,

$$\psi_0(r_1, \theta_0) \leqq \psi_0(r_2, \theta_0), \qquad (0 \leqq r_1 < r_2 \text{ on } S), \tag{7}$$

the sign of equality holding if and only if $K \equiv 0$ on $\theta = \theta_0$.

By (5) and l'Hospital's rule we have

$$\lim_{r\to 0} \psi_1(r) = \lim_{r\to 0} \frac{\int_0^{2\pi} \mu(r, \theta)\, d\theta}{2\pi r} = \lim_{r\to 0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \mu(r, \theta)}{\partial r}\, d\theta = 1,$$

so that $\psi_1(r)$ is continuous for $r \geqq 0$ on S. Also, $\psi_1(r)$ is monotonic non-decreasing, since by (7) we have

$$\psi_1(r_1) = \frac{\int_0^{2\pi} \mu(r_1, \theta)d\theta}{2\pi r_1} = \frac{1}{2\pi}\int_0^{2\pi} \psi_0(r_1, \theta)d\theta \leqq \frac{1}{2\pi}\int_0^{2\pi} \psi_0(r_2, \theta)d\theta = \psi_1(r_2), \qquad (0 < r_1 < r_2 \text{ on } S), \tag{8}$$

the sign of equality holding if and only if $K \equiv 0$ on S. Thus $\psi_1(r)$ satisfies Condition B.

We note that (8) could have been obtained equally well by differentiation. By (5) and the convexity of $\mu(r, \theta_0)$, we have $\mu(r, \theta) > 0$ for $r > 0$ on S, so that $\partial\mu/\partial r$ exists for $r > 0$. By (6), we have

$$2\pi r^2 \frac{d\psi_1(r)}{dr} = r\frac{dl(r)}{dr} - l(r) = \int_0^{2\pi} \left[r\frac{\partial\mu(r, \theta)}{\partial r} - \mu(r, \theta) \right] d\theta \geqq 0, \tag{9}$$

the sign of equality holding for $r > 0$ on S if and only if $K \equiv 0$ on S.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then $l(r)$ is a monotonic increasing convex function of r and satisfies the inequality*

$$l(r) \geqq 2\pi r; \tag{10}$$

$l(r)$ is strictly convex and satisfies the strict inequality for $r > 0$ on S if S is not a developable surface, and is linear and satisfies the equality if S is developable.

The corollary follows from Theorem 3 and the identity

$$l(r) \equiv 2\pi r + \phi_1(r).$$

9. Area theorems. On surfaces of non-positive Gaussian curvature, the area function,

$$a(r) \equiv \int_0^r \int_0^{2\pi} \mu(\rho, \theta)d\rho d\theta,$$

has properties similar to those given for $l(r)$.

THEOREM 4. *Let S be an analytic surface of non-positive Gaussian curvature, and let $a(r)$ denote the area of the geodesic circle on S with fixed center P_0 and geodesic radius r. Then the function*

$$\phi_2(r) \equiv a(r) - \pi r^2, \qquad (r \geqq 0 \text{ on } S),$$

satisfies Condition A, and the function

$$\psi_2(r) \equiv \frac{a(r)}{\pi r^2}, \qquad (r > 0 \text{ on } S),$$

$$\psi_2(0) \equiv 1$$

satisfies Condition B.

Proof. We have

$$\phi_2(r) = \int_0^r \phi_1(\rho)d\rho,$$

whence

$$\phi_2'(r) = \phi_1(r) \geqq 0, \qquad \phi_2''(r) = \phi_1'(r) \geqq 0,$$

the signs of equality holding for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_2(0)=0$, the results indicated for $\phi_2(r)$ follow.

The continuity of $\psi_2(r)$ at $r=0$ follows from (5) and l'Hospital's rule. Since $l(r)$ is convex and satisfies $l(0)=0$, we have

$$l(\rho_1) \leqq \frac{\rho_1}{\rho_2} l(\rho_2), \qquad (0 < \rho_1 < \rho_2 \text{ on } S);$$

the sign of equality holds if and only if $K\equiv 0$ on S. Hence

$$\begin{aligned} a(r_1) &= \int_0^{r_1} l(\sigma)d\sigma = \int_0^{r_2} l\left(\frac{r_1}{r_2}\,\rho\right)\frac{r_1}{r_2}\,d\rho \\ &\leqq \int_0^{r_2} l(\rho)\,\frac{r_1^2}{r_2^2}\,d\rho = a(r_2)\,\frac{r_1^2}{r_2^2}, \qquad (0 < r_1 < r_2 \text{ on } S), \end{aligned}$$

whence

$$\psi_2(r_1) \leqq \psi_2(r_2), \qquad (0 \leqq r_1 < r_2 \text{ on } S), \tag{11}$$

the sign of equality holding if and only if $K\equiv 0$ on S. This completes the proof of the theorem.

We note that (11) could have been obtained equally well by differentiation. We have

$$\begin{aligned} \pi r^3 \frac{d\psi_2(r)}{dr} &= rl(r) - 2a(r) = rl(r) - 2\int_0^r l(\rho)d\rho \\ &\geqq rl(r) - 2\int_0^r \frac{\rho}{r}\,l(r)d\rho = 0, \qquad (r > 0 \text{ on } S), \end{aligned} \tag{12}$$

the sign of equality holding if and only if $K\equiv 0$ on S.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then $a(r)$ is a monotonic increasing strictly convex function of r and satisfies the inequality*

$$a(r) \geqq \tfrac{4}{3}\pi r^2,$$

the sign of equality holding for $r>0$ on S if and only if S is a developable surface.

The corollary follows from Theorem 4 and the identity

$$a(r) \equiv \pi r^2 + \phi_2(r).$$

Actually, $a(r)$ and the function $\phi_2(r) \equiv a(r) - \pi r^2$, which we have shown to be non-negative, satisfy even stronger convexity conditions than those given in Theorem 4 and its corollary, as we now shall show. For a non-negative function $g(r)$, the convexity of $[g(r)]^{1/2}$ implies the convexity of $g(r)$; but the converse does not hold. The proof of Theorem 5 will be shortened by use of the following lemma showing that certain conditions which clearly imply the convexity of $g(r)$ also imply the convexity of $[g(r)]^{1/2}$.

LEMMA 2. *If $g(r)$ is a non-negative function for which $g'''(r)$ exists in the interval $\alpha \leqq r < \beta$, satisfying*

$$h(\alpha) \equiv 2g(\alpha)g''(\alpha) - [g'(\alpha)]^2 \geqq 0$$

and

$$g'''(r) \geqq 0, \qquad (\alpha \leqq r < \beta),$$

then $[g(r)]^{1/2}$ is convex in $\alpha \leqq r < \beta$, and is strictly convex there provided

$$g'''(r) > 0, \qquad (\alpha < r < \beta).$$

Proof. If we let

$$f(r) \equiv [g(r)]^{1/2},$$

then at points where $f(r) \neq 0$ we have

$$f'' = \tfrac{1}{4}g^{-3/2}(2gg'' - g'^2) = \tfrac{1}{4}g^{-3/2}h.$$

Now

$$h' = 2gg''', \tag{13}$$

so that from the hypotheses we get $h(\alpha) \geqq 0$, $h'(r) \geqq 0$, $\alpha \leqq r < \beta$, whence $h(r) \geqq 0$. Then $f''(r) \geqq 0$ at points where $f(r) \neq 0$. Since, further, the non-negative function $f(r)$ satisfies the convexity inequality (3) for points where $f(r) = 0$, it follows that the continuous function $f(r)$ is convex for $\alpha \leqq r < \beta$.

If $g'''(t) > 0$, $\alpha < r < \beta$, then the non-negative convex function $g(r)$ can vanish at no more than one point of $\alpha \leqq r < \beta$, whence, by (13), we have $h(r) > 0$, $\alpha < r < \beta$. It follows that we have $f''(r) > 0$ except at most at one point of $\alpha < r < \beta$, so that $f(r)$ is strictly convex for $\alpha \leqq r < \beta$.

THEOREM 5. *Let S be a surface of non-positive Gaussian curvature, and let $a(r)$ denote the area of the geodesic circle on S with fixed center P and geodesic radius r. Then*

$$f(r) \equiv [a(r)]^{1/2}$$

is a convex function of r for $r \geqq 0$ on S; $f(r)$ is strictly convex there if S is not a developable surface, and is linear if S is developable. Further, the function

$$[\phi_2(r)]^{1/2} \equiv [a(r) - \pi r^2]^{1/2}$$

satisfies Condition A.

Proof. We have

$$a'(r) = l(r) = \int_0^{2\pi} \mu(r, \theta) d\theta, \qquad a''(r) = \int_0^{2\pi} \frac{\partial \mu(r, \theta)}{\partial r} d\theta,$$

$$a'''(r) = \int_0^{2\pi} \frac{\partial^2 \mu(r, \theta)}{\partial r^2} d\theta,$$

whence, in addition to $a(0) = 0$, we have

$$a'(0) = 0, \qquad a'''(r) \geqq 0, \qquad (r \geqq 0 \text{ on } S),$$

with $a'''(r) > 0$ for $r > 0$ on S unless $K \equiv 0$ on S. Then for $0 \leqq r$ on S, $a(r)$ satisfies the hypotheses concerning $g(r)$ in Lemma 2, so that $[a(r)]^{1/2}$ is convex for $r \geqq 0$ on S, and is strictly convex there if S is not a developable surface. If S is developable, then $[a(r)]^{1/2} = \pi^{1/2} r$, so that $[a(r)]^{1/2}$ is linear.

Similarly we have

$$\phi_2'(r) = l(r) - 2\pi r = \phi_1(r), \qquad \phi_2''(r) = \phi_1'(r), \qquad \phi_2'''(r) = \phi_1''(r),$$

whence, in addition to $\phi_2(0) = 0$, we have

$$\phi_2'(0) = 0, \qquad \phi_2'''(r) \geqq 0, \qquad (r \geqq 0 \text{ on } S),$$

with $\phi_2'''(r) > 0$ for $r > 0$ on S unless $K \equiv 0$ on S. It follows from Lemma 2 and Theorem 4 that $[\phi_2(r)]^{1/2}$ satisfies Condition A.

10. The isoperimetric inequality. As was pointed out in §4, the isoperimetric inequality (4) holds for all Jordan regions on surfaces S of non-positive Gaussian curvature, and characterizes these surfaces, so that in particular we have

$$a(r) \leqq \frac{1}{4\pi} [l(r)]^2 \tag{14}$$

for all geodesic circles on these surfaces. In this section we shall include independent proofs of (14).

THEOREM 6. *Let S be an analytic surface of non-positive Gaussian curvature, and $l(r)$ denote the length of the circumference, and $a(r)$ the area, of the geodesic circle on S with fixed center P_0 and geodesic radius r. Then the functions*

$$\phi_3(r) \equiv \frac{1}{4\pi} (2\pi r)[l(r)] - a(r) \equiv \tfrac{1}{2} r l(r) - a(r)$$

and

$$\phi_4(r) \equiv \frac{1}{4\pi}[l(r)]^2 - a(r), \qquad (r \geqq 0 \text{ on } S),$$

satisfy Condition A, and the function

$$\psi_4(r) \equiv \frac{\frac{1}{4\pi}[l(r)]^2}{a(r)}, \qquad (r > 0 \text{ on } S),$$
$$\psi_4(0) \equiv 1$$

satisfies Condition B.

Proof. We have

$$\phi_3'(r) = \tfrac{1}{2}[rl'(r) - l(r)] \geqq 0$$

by (9), and

$$\phi_3''(r) = \tfrac{1}{2}rl''(r) \geqq 0$$

by the convexity of $l(r)$; the signs of equality holds for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_3(0)=0$, it follows that $\phi_3(r)$ satisfies Condition A.

Again, we have

$$\phi_4'(r) = l(r)\left[\frac{1}{2\pi}l'(r) - 1\right] = l(r)\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\mu(r,\theta)}{\partial r}\,d\theta - 1\right] \geqq 0 \tag{15}$$

by (6); and

$$\phi_4''(r) = \frac{1}{2\pi}l(r)l''(r) + l'(r)\left[\frac{1}{2\pi}l'(r) - 1\right] \geqq 0;$$

the signs of equality hold for $r>0$ on S if and only if $K\equiv 0$ on S. Since $\phi_4(0)=0$, it follows that $\phi_4(r)$ satisfies Condition A.

Now $\psi_4(r)$ is continuous at $r=0$ by (5) and l'Hospital's rule. We have

$$[a(r)]^2\psi_4'(r) = \frac{1}{4\pi}l(r)p(r),$$

where

$$p(r) = 2a(r)l'(r) - [l(r)]^2.$$

Since $p(0)=0$ and

$$p'(r) = 2a(r)l''(r) \geqq 0, \qquad (r > 0 \text{ on } S),$$

we have $p(r)\geqq 0$, whence $\psi_4'(r)\geqq 0$, $r>0$ on S; the sign of equality holds if and only if $K\equiv 0$ on S. Hence $\psi_4(r)$ satisfies Condition B.

COROLLARY. *If S is an analytic surface of non-positive Gaussian curvature, then*

$$a(r) \leqq \frac{1}{4\pi}(2\pi r)[l(r)],$$

and a fortiori

$$a(r) \leqq \frac{1}{4\pi}[l(r)]^2, \qquad (r \geqq 0 \text{ on } S),$$

the signs of equality holding for $r>0$ on S if and only if S is a developable surface.

The corollary follows from Theorem 6 and the identities

$$a(r) \equiv \frac{1}{4\pi}(2\pi r)[l(r)] - \phi_3(r),$$

$$a(r) \equiv \frac{1}{4\pi}[l(r)]^2 - \phi_4(r).$$

The corollary follows equally well from (10) and (12).

11. Geodesic circular sectors. The convexity relations derived in §§8, 9, and 10 for geodesic circles depend on convexity relations which hold on each geodesic radius. The integration with respect to θ was taken between limits 0 and 2π for geometric reasons only. We might confine our attention equally well to a *geodesic circular sector* $0 \leqq r < r_0$, $\theta_1 \leqq \theta \leqq \theta_2$, with pole $r=0$ at P_0, and fixed angle (θ_1, θ_2), on S. Equations (1) and (5) still hold, but we do not assume that there necessarily is an entire geodesic circle $0 \leqq r < r_0$, $0 \leqq \theta < 2\pi$, with center at P_0, on S.

For a given surface S, a given pole P_0 of geodesic polar coördinates on S, and a given angle (θ_1, θ_2), we let $l(r; \theta_1, \theta_2)$ denote the length of the bounding arc of geodesic circular sector,

$$l(r; \theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \mu(r, \theta)\, d\theta,$$

and $a(r; \theta_1, \theta_2)$ the area,

$$a(r; \theta_1, \theta_2) = \int_0^r \int_{\theta_1}^{\theta_2} \mu(\rho, \theta)\, d\rho d\theta = \int_0^r l(\rho; \theta_1, \theta_2)\, d\rho,$$

of the geodesic circular sector of geodesic radius r, and angle from θ_1 to θ_2.

We have the following results.

THEOREM 7. *Let S be an analytic surface of non-positive Gaussian curvature, and let $l(r; \theta_1, \theta_2)$ denote the length of the bounding arc of geodesic circular sector, and $a(r; \theta_1, \theta_2)$ the area, of the geodesic circular sector on S with fixed pole P_0, fixed angle from θ_1 to θ_2, $\theta_1 < \theta_2$, and geodesic radius r. Then the functions*

$$\phi_5(r) \equiv l(r;\theta_1,\theta_2) - (\theta_2-\theta_1)r,$$

$$\phi_6(r) \equiv a(r;\theta_1,\theta_2) - \left(\frac{\theta_2-\theta_1}{2}\right)r^2,$$

$$\phi_7(r) \equiv \frac{[(\theta_2-\theta_1)r][l(r;\theta_1,\theta_2)]}{2(\theta_2-\theta_1)} - a(r;\theta_1,\theta_2) \equiv \tfrac{1}{2}rl(r;\theta_1,\theta_2) - a(r;\theta_1,\theta_2),$$

and

$$\phi_8(r) \equiv \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)} - a(r;\theta_1,\theta_2), \qquad (r \geqq 0, \theta_1 \leqq \theta \leqq \theta_2, \text{ on } S),$$

satisfy Condition A; and the functions

$$\psi_5(r) \equiv \frac{l(r;\theta_1,\theta_2)}{(\theta_2-\theta_1)r},$$

$$\psi_6(r) \equiv \frac{a(r;\theta_1,\theta_2)}{\left(\frac{\theta_2-\theta_1}{2}\right)r^2},$$

$$\psi_8(r) \equiv \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)[a(r;\theta_1,\theta_2)]}, \qquad (r > 0, \theta_1 \leqq \theta \leqq \theta_2, \text{ on } S),$$

$$\psi_j(0) \equiv 1, \qquad (j = 5, 6, 8),$$

satisfy Condition B.

The proofs are similar to the proofs of analogous results given in §§8, 9, and 10, and will not be given here.

It follows as a corollary to Theorem 7 that if S is a surface of non-positive Gaussian curvature, then $l(r;\theta_1,\theta_2)$ and $a(r;\theta_1,\theta_2)$ are convex functions which satisfy the inequalities

$$l(r;\theta_1,\theta_2) \geqq (\theta_2-\theta_1)r,$$

$$a(r;\theta_1,\theta_2) \geqq \left(\frac{\theta_2-\theta_1}{2}\right)r^2,$$

and

$$a(r;\theta_1,\theta_2) \leqq \tfrac{1}{2}rl(r;\theta_1,\theta_2) \leqq \frac{[l(r;\theta_1,\theta_2)]^2}{2(\theta_2-\theta_1)}, \qquad (\theta_1 < \theta_2);$$

the signs of equality hold for $r>0$ on S if and only if S is a developable surface.

Similarly, Theorem 5 also generalizes to geodesic circular sectors on surfaces of non-positive Gaussian curvature, with $[a(r;\theta_1,\theta_2)]^{1/2}$ and $[\phi_6(r)]^{1/2}$ in place of $[a(r)]^{1/2}$ and $[\phi_2(r)]^{1/2}$, respectively.

In place of the sector $\theta_1 \leqq \theta \leqq \theta_2$, we might consider an arbitrary measurable set of values of θ, but shall not pursue this further.

12. On regular surfaces of non-negative Gaussian curvature. The above results concerning surfaces of non-positive Gaussian curvature hold in the large and are unaffected by singular points. The following rather analogous results for surfaces of non-negative curvature hold in general only on parts of S where there are no singular points of the surface, or of the family of geodesics, other than at the pole of geodesic polar coördinates; and some of the results hold only in the small even where there are no singular points.

A function $f(x)$ is said to be *concave* provided $-f(x)$ is convex.

A surface S given in terms of geodesic coördinates, or in terms of geodesic polar coördinates, will be said to be *regular* provided there are no singular points on S except, in the case of geodesic polar coördinates, at the pole P_0.

Lemma 1 holds if we add the restriction that S is regular, and replace "non-positive" by "non-negative," and "convex" by "concave." Theorems 1 and 2 hold with the same alterations, and the corollary of Theorem 1 follows *mutatis mutandis*.

We alter Conditions A and B to apply to a "regular surface S of non-negative Gaussian curvature," changing "non-decreasing," "convex," and "increasing" to "non-increasing," "concave," and "decreasing," respectively. The altered conditions will be denoted Conditions A^* and B^*, respectively.

Then *the functions* $\phi_j(r)$, $j=1, 2, 3$, *satisfy Condition* A^*, *and the functions* $\psi_j(r)$, $j=1, 2, 4$, *satisfy Condition* B^*, *on regular analytic surfaces of non-negative Gaussian curvature.*

By the expression for $\phi_4'(r)$ in (15), $\phi_4(r)$ *is monotonic non-increasing on any regular analytic surface* S *of non-negative Gaussian curvature, and is monotonic decreasing if* S *is not developable.* Actually, since $\phi_4(r)$ is analytic, it follows from $\phi_4(0)=\phi_4'(0)=0$ and $\phi_4(r)\leqq 0$ that there is an $r_0=r_0(S, P_0)$ such that $\phi_4(r)$ is concave for $0\leqq r\leqq r_0$. A consideration of the sphere shows, however, that $\phi_4(r)$ is not necessarily concave for all $r>0$ on S.

The properties of the function $\phi_j(r)$ yield the following results concerning $l(r)$ and $a(r)$.

We have

$$l(r) = 2\pi r + \phi_1(r),$$
$$l'(r) = 2\pi + \phi_1'(r),$$

and

$$l''(r) = \phi_1''(r).$$

Since on regular surfaces of non-negative Gaussian curvature the function $\phi_1(r)$ satisfies Condition A^*, on these surfaces we have

$$\phi_1(r) \leqq 0, \qquad \phi_1''(r) \leqq 0, \qquad (r \geqq 0 \text{ on } S).$$

It follows that *on regular analytic surfaces of non-negative Gaussian curvature the function* $l(r)$ *is concave and satisfies*

$$l(r) \leqq 2\pi r;$$

$l(r)$ is strictly concave and satisfies the strict inequality for $r>0$ on S if S is not a developable surface. Also, on these surfaces we have $\phi_1'(0)=0$, so that, since $\phi_1(r)$ is concave, *for a given regular analytic surface S of non-negative Gaussian curvature and for a given pole P_0 on S, either $l(r)$ is monotonic increasing on S or there is an $r_0=r_0(S, P_0)>0$ such that $l(r)$ is monotonic increasing for $0\leqq r\leqq r_0$ and monotonic decreasing for $r\geqq r_0$ on S.*

Again, we have

$$a(r) = \pi r^2 + \phi_2(r),$$
$$a'(r) = 2\pi r + \phi_2'(r) = 2\pi r + \phi_1(r) = l(r),$$

and

$$a''(r) = 2\pi + \phi_2''(r) = 2\pi + \phi_1'(r) = l'(r).$$

On regular analytic surfaces of non-negative Gaussian curvature we have

$$\phi_2(r) \leqq 0, \qquad \phi_2''(0) = 0, \qquad \phi_2''(r) \leqq 0, \qquad (r \geqq 0 \text{ on } S).$$

Hence *on regular analytic surfaces of non-negative Gaussian curvature, $a(r)$ satisfies*

$$a(r) \leqq \pi r^2;$$

the strict inequality holds for $r>0$ on S if S is not a developable surface. Further, for a given regular analytic surface S of non-negative Gaussian curvature, and for a given pole P_0 on S, either $a(r)$ is strictly convex, or there is an $r_0=r_0(S, P_0)>0$ such that $a(r)$ is strictly convex for $0\leqq r\leqq r_0$ and strictly concave for $r\geqq r_0$ on S.

The interval $0\leqq r\leqq r_0$ on which $a(r)$ is convex coincides with the interval on which $l(r)$ is increasing.

On a sphere, for example, $l(r)$ is increasing and concave, and $a(r)$ is convex, until a hemisphere has been covered. Thereafter, $l(r)$ is decreasing but still concave, and $a(r)$ becomes concave, until the other hemisphere has been covered. At the point diametrically opposite the pole we encounter a singular point of the geodesic representation.

If K_0 is the Gaussian curvature of a surface S at a regular point P_0, then the area and the length of circumference of the geodesic circle with center P_0 and radius r on S are given [2, p. 209] by

$$a(r) = \pi r^2 - \tfrac{1}{12}\pi K_0 r^4 + \cdots,$$

and

$$l(r) = 2\pi r - \tfrac{1}{3}\pi K_0 r^4 + \cdots,$$

respectively, so that

$$a(r) - \frac{1}{4\pi}[l(r)]^2 = \tfrac{1}{4}\pi K_0 r^4 + \cdots.$$

It follows that if $K_0>0$, then *for sufficiently small $r>0$* we have the isoperimetric

inequality with sign reversed:

$$a(r) > \frac{1}{4\pi}[l(r)]^2.$$

We now improve the above result, as follows.

Since $\phi_3(0) = \phi_4(0) = 0$, it follows from the properties of $\phi_3(r)$ and $\phi_4(r)$ given earlier in this section that *on regular surfaces of non-negative Gaussian curvature we have*

$$a(r) \geqq \frac{1}{4\pi}(2\pi r)[l(r)] \geqq \frac{1}{4\pi}[l(r)]^2,$$

the signs of equality holding for $r>0$ on S if and only if S is a developable surface.

The functions related to geodesic circular sectors, which we introduced in §11, have, on regular surfaces of non-negative Gaussian curvature, properties analogous to those of the corresponding functions related to geodesic circles on surfaces of the same class.

13. Converse questions. In all the above analyses we have assumed that the Gaussian curvature of S either is non-positive or is non-negative. In most instances we have obtained, in the two cases, conclusions which are distinct except for the dividing class of developable surfaces. Thus by logical exclusion we obtain several characterizations of the above classes of surfaces.

For instance, a regular analytic surface S is a surface of non-negative Gaussian curvature, but not a developable surface, if and only if for each pole P_0 on S we have

$$l(r) < 2\pi r \tag{16}$$

for all $r>0$ on S.

We have shown that the condition $K \geqq 0$, $K \not\equiv 0$, on S implies (16). Conversely, if we should have $K_0 < 0$ at some P_0 on S, then we would have $K<0$ through some neighborhood of P_0, and therefore, in the neighborhood, we would have

$$l(r) > 2\pi r;$$

also, if we should have $K \equiv 0$ on S, then we would have

$$l(r) \equiv 2\pi r;$$

hence (16) implies $K \geqq 0$, $K \not\equiv 0$, on S.

We might replace (16) by

$$a(r) < \pi r^2,$$

and might give similar characterizations of surfaces with $K \leqq 0$, $K \not\equiv 0$, and of surfaces with $K \equiv 0$.

In the same way we have the following result.

THEOREM 8. *The regular analytic surface S is, i) a surface of non-positive Gaussian curvature, but not developable, ii) a surface of non-negative Gaussian curvature, but not developable, or, iii) a developable surface, if and only if we have*

$$(17)\quad i)\ \phi_j(r) > 0,\quad ii)\ \phi_j(r) < 0,\quad or\quad iii)\ \phi_j(r) \equiv 0,\quad (j = 1, 2, \cdots, 7, or\ 8),$$

respectively, for all poles P_0 and all $r>0$ on S.

We might replace (17) by the convexity conditions

$$i)\ \phi_j''(r) > 0,\quad ii)\ \phi_j''(r) < 0,\quad or\quad iii)\ \phi_j''(r) \equiv 0,\quad (j = 1, 2, 3, 5, 6, or\ 7),$$

and so on.

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SOME THEOREMS ON CYCLIC POLYGONS INSCRIBED IN A CIRCLE*

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1. Introduction. The Lemoine and Brocard points, as well as their diverse properties, are well known to whoever has studied the geometry of the triangle.

In the present paper we shall show that these points, with their inherent properties, are encountered not only in the triangle but, on the contrary, in all figures which are obtained by joining in pairs the corresponding points of a cyclic projectivity, or more generally of any projectivity on a circle.

The case of the triangle will appear as a particular case if one takes care to consider it as the figure which is obtained by means of the same process, since this gives rise on a circle to a cyclic projectivity of the third order. In this case the center and the axis of collineation are respectively the Lemoine point and line. It goes without saying that in the slightly more general case where the figure is no longer a proper triangle, but a set of any three chords of a circle, the same remarks may be made.

2. Definitions and preliminaries. Let us take, on a conic C, a set of points

* Translated from the French by Howard Eves.

$A_1, A_2, \cdots, A_n$ belonging to a cyclic projectivity of order n. The polygon which is obtained by joining by lines each of these points with its corresponding point (continuing the process until one arrives at the initial point), will be called a *cyclic n-gon.*

The cyclic n-gons will be of the *first, second, third,* $\cdots$ *kind* if each of their sides joins points between which there are no, one, two, $\cdots$ points of the set.

It is evident that if n is odd each set of n points defines $(n-1)/2$ n-gons, while if n is even each set of n points defines $n/2$ n-gons. [1].

On the other hand since the two point sets $(A_1, A_2, \cdots, A_n)$, $(A_2, A_3, \cdots, A_1)$ are projective, the lines $A_1A_2, A_2A_3, \cdots, A_nA_1$ envelop a conic C_1 which has double contact with the conic C. Similarly, the two point sets $(A_1, A_2, A_3, \cdots, A_n)$, $(A_3, A_4, A_5, \cdots, A_2)$ are also projective and consequently the lines $A_1A_3, A_2A_4, \cdots, A_nA_2$ envelop another conic C_2 which has double contact with the conic C, and so on. Thus:

The cyclic n-gons of the first, second, third, $\cdots$ *kind which are inscribed in the same conic C, envelop conics* $C_1, C_2, C_3, \cdots$ *which have double contact with the conic C.*

But since two conics C and C_1, which have double contact, can be projected into two concentric circles, it follows that:

All cyclic n-gons inscribed in a conic may be considered as projections of a regular polygon.

But the circles inscribed in the regular polygons of the first, second, third, $\cdots$ kind are concentric, and thus are tangent to each other and to the circumcircle at the circular points at infinity. Consequently:

The conics $C_1, C_2, \cdots$ *are tangent to C at the same points.*

3. Lemoine point and line. Concerning cyclic n-gons derived from the same cyclic projectivity we shall now establish

THEOREM 1. *All the cyclic n-gons of all kinds, which are inscribed in the same circle, have the same Lemoine point and line.* [2]

First of all we notice that all the cyclic n-gons of all kinds have the same center and axis of collineation.

We shall show that the center and axis of collineation are respectively the Lemoine point and line.

Consider on the circle any n-gon of the first kind, $A_1A_2 \cdots A_n$. The two point sets $(A_1, A_2, A_3, \cdots, A_{n-2}, A_{n-1}, A_n)$, $(A_2, A_3, A_4, \cdots, A_{n-1}, A_n, A_1)$ are projective, and consequently the lines $A_1A_1, A_2A_n, A_3A_{n-1}, \cdots$ pass through a common point M, situated on the axis of collineation of the projectivity. The polar, then, of point M, which will pass through the vertex A_1, will also pass through the center of collineation of the projectivity. But this polar forms a harmonic set with the lines A_1A_1, A_1A_2, A_1A_n. It is then a symmedian of triangle $A_nA_1A_2$ and it follows that for each of its points, the ratio of the distances to the sides A_nA_1, A_1A_2 is equal to the ratio of the segments (A_nA_1), (A_1A_2).

Similarly, the polar of the point of intersection of the lines A_2A_2, A_1A_3, $A_4A_n \cdots$ passes through the point A_3 and also through the center of collineation, and is a symmedian of triangle $A_1A_2A_3$, whence each of its points has the ratio of its distances from the sides A_1A_2, A_2A_3 equal to that of the segments (A_1A_2), (A_2A_3). By a similar treatment of the other vertices A_3, A_4, $\cdots$, A_n, the theorem readily follows for cyclic n-gons of the first kind.

The same argument holds for cyclic polygons of other kinds.

4. Brocard points. In regard to Brocard points we shall prove

THEOREM 2. *All the cyclic n-gons of the same kind, inscribed in the same circle, have the same Brocard points.* [3]

Let $A_1A_2A_3 \cdots A_n$ be any cyclic n-gon of the first kind inscribed in a circle, and let K be the Lemoine point. From the center O of the circumcircle let us drop perpendiculars onto the sides A_1A_2, A_2A_3, A_3A_4, $\cdots$, A_nA_1 of the n-gon. These cut the parallels to the sides of the polygon drawn through point K in the points L_1, L_2, L_3, $\cdots$, L_n respectively. The isosceles triangles $A_1L_1A_2$, $A_2L_2A_3$, $\cdots$ are similar. The circle having the segment OK as diameter obviously passes through the points L_1, L_2, $\cdots$, L_n. Let Ω_1 be the point of intersection of the line A_1L_1 with the circle on OK as diameter. We have

$$\begin{aligned} \sphericalangle KO\Omega_1 &= 180^\circ - \sphericalangle KL_1\Omega_1 = \sphericalangle KL_1A_1 \\ &= \sphericalangle A_2A_1L_1 = \omega = \sphericalangle A_3A_2L_2 = \sphericalangle KL_2A_2. \end{aligned}$$

From this we conclude that the line A_2L_2 passes through Ω_1. Similarly we would find that the lines A_3L_3, A_4L_4, $\cdots$, A_nL_n all pass through the same point Ω_1. The lines A_1L_n, A_2L_1, A_3L_2, $\cdots$, in their turn, all pass through a point Ω_2 of the circle on OK as diameter.

The points Ω_1 and Ω_2 are evidently symmetric with respect to the line OK.

COROLLARY 1. *All the cyclic n-gons of all kinds inscribed in the same circle have the same Brocard circle.*

COROLLARY 2. *The Brocard points and the vertices of the isosceles triangles of all the cyclic n-gons of all kinds inscribed in the same circle, lie on the Brocard circle. Also the circles of the two groups of n adjoint circles* [4] *for each n-gon intersect on the Brocard circle.*

It is known that if a side of an angle, of constant magnitude ω, passes through a fixed point Ω, while the vertex of the angle lies on a circle, the other side will envelop a conic which will have double contact with the circle in question, and for which one of the foci is the point Ω.*

COROLLARY 3. *The foci of the conic which envelops all the cyclic n-gons of the same kind, inscribed in the same circle, coincide with the Brocard points of those n-gons.*

* Poncelet, Applications d'analyse et de géométrie (1817), t. II, p. 462.

COROLLARY 4. *The projections of the Brocard points on the sides of all the cyclic n-gons of the same kind, incribed in the same circle, are found on the principal circle of the conic which envelops the n-gons in question.*

5. Calculation of the angle ω. Let $x_1, x_2, \cdots x_n$ be the distances of the point K from the sides A_1A_2, A_2A_3, $\cdots$, A_nA_1, which have lengths $a_1, a_2, \cdots, a_n$ respectively, of a cyclic n-gon of the first kind. We have

$$x_1/a_1 = x_2/a_2 = \cdots = x_n/a_n = \tfrac{1}{2}\tan\omega,$$

or

$$a_1x_1/a_1^2 = a_2x_2/a_2^2 = \cdots = a_nx_n/a_n^2 = \tfrac{1}{2}\tan\omega = 2S/\sum a_i^2,$$

where S is the area of the n-gon. Therefore

$$\tan\omega = 4S/\sum a_i^2 = \sum a_i(4R^2 - a_i^2)^{1/2}/\sum a_i^2,$$

where R is the radius of the circumcircle.

It follows that the quantity $S/\sum a_i^2$ is the same for all the cyclic n-gons of the same kind which are inscribed in the same circle.

6. First Lemoine circle. The generalization to cyclic n-gons of the first Lemoine circle of a triangle is given by the following:

THEOREM 3. *If through the Lemoine point K of an inscriptible cyclic n-gon lines are drawn parallel to the sides, the points of intersection of these parallels with the sides adjacent to their corresponding sides lie on a circle.*

Let A_n, A_1, A_2, A_3 be four successive vertices of the inscribed cyclic n-gon, and let M_1 on A_nA_1, M_2 on A_2A_3, M_3 and M_4 on A_1A_2 be such that KM_1 and KM_2 are parallel to A_1A_2, KM_3 to A_2A_3, and KM_4 to A_nA_1.

Since the line A_2K is a symmedian of triangle $A_1A_2A_3$, the line M_2M_3 will be antiparallel to the side A_1A_3, and consequently will be perpendicular to the line OA_2 as well as to the line LP, L and P being the midpoints of the segments OK and KA_2. But since P is also midpoint of the segment M_2M_3, we will have $LM_2 = LM_3$. For the same reason we will have $LM_1 = LM_4$. But the trapezoid $M_1M_2M_3M_4$ is inscriptible in a circle, since

$$\sphericalangle M_2M_3M_4 = \sphericalangle A_1A_3A_2 = \sphericalangle A_1A_nA_2 = \sphericalangle M_1M_4M_3.$$

The center of this circle is the point L.

If we consider the vertices A_1, A_2, A_3, A_4, the corresponding trapezoid will also be inscriptible in a circle which will have the points M_2, M_4 and the center L in common with the circumcircle of the preceding trapezoid. Continuing in this manner around the n-gon, the theorem is established.

7. Some formulas. Let θ be the angle $A_1A_2A_3$, $x_n, x_1, x_2, \cdots, x_{n-1}$ the distances of point K from the side A_nA_1, A_1A_2, $A_2A_3, \cdots, A_{n-1}A_n$, and $a_n, a_1, a_2, \cdots, a_{n-1}$ the lengths of these sides. As above we have

$$x_1/a_1 = x_2/a_2 = \cdots = \tfrac{1}{2}\tan\omega = 2S/\sum a_i^2.$$

On the other hand we have

$$x_1 = (KM_3)\sin\theta = \tfrac{1}{2}a_1\tan\omega, \qquad x_2 = (KM_2)\sin\theta = \tfrac{1}{2}a_2\tan\omega,$$

and

$$\begin{aligned}(M_2M_3)^2 &= 4(PM_2)^2 = (KM_3)^2 + (KM_2)^2 - 2(KM_3)(KM_2)\cos\theta \\ &= \tfrac{1}{4}(a_1^2 + a_2^2 - 2a_1a_2\cos\theta)\tan^2\omega\csc^2\theta \\ &= \tfrac{1}{4}(A_2A_3)^2\tan^2\omega\csc^2\theta = R^2\tan^2\omega.\end{aligned}$$

Consequently

$$\rho^2 = \tfrac{1}{4}(R^2 + R^2\tan^2\omega) = \tfrac{1}{4}R^2\sec^2\omega,$$

or

$$\rho = \tfrac{1}{2}R\sec\omega.$$

The length s of the antiparallel (M_2M_3) is

$$s = 4SR/\sum a_i^2 = R\tan\omega.$$

This shows that all these antiparallels are equal.

For the distance $(OK) = d$ we have

$$\begin{aligned}d^2 &= 4(LK)^2 = 4(LM_2)^2 + 4(KM_2)^2 - 4(KM_2)(M_1M_2) \\ &= R^2\sec^2\omega + a_2^2\tan^2\omega\csc^2\theta - 2a_2\tan\omega\csc\theta(x_n\csc\phi + x_2\csc\theta),\end{aligned}$$

where $\phi = \sphericalangle KM_1A_1$. But

$$2x_2 = a_2\tan\omega = 2R\sin\theta_2\tan\omega, \qquad 2x_n = a_n\tan\omega = 2R\sin\theta_n\tan\omega,$$

where $\theta_2 = \sphericalangle A_2A_1A_3$ and $\theta_n = \sphericalangle A_1A_2A_n$. Therefore

$$\begin{aligned}d^2 &= R^2\sec^2\omega - 4R^2\tan^2\omega\sin\theta_2\sin\theta_n\csc\theta\csc\phi \\ &= R^2\sec^2\omega - 4R^2\tan^2\omega\sin\theta_2\sin\theta_n\csc(\theta_1+\theta_2)\csc(\theta_1+\theta_n),\end{aligned}$$

where $\theta_1 = \sphericalangle A_1A_3A_2 = \sphericalangle A_1A_nA_2$.

Let $1/\lambda$ represent the anharmonic ratio of the four successive vertices A_n, A_1, A_2, A_3. Then

$$\begin{aligned}(A_nA_3A_1A_2) = 1 - \lambda &= \sin\tfrac{1}{2}(A_nA_1)\sin\tfrac{1}{2}(A_2A_3)\csc\tfrac{1}{2}(A_1A_3)\csc\tfrac{1}{2}(A_nA_2) \\ &= \sin\theta_2\sin\theta_n\csc(\theta_1+\theta_2)\csc(\theta_1+\theta_n),\end{aligned}$$

and we have

$$d^2 = R^2\sec^2\omega - 4R^2(1-\lambda)\tan^2\omega = R^2 - (3-4\lambda)s^2.$$

Finally, we have

$$(O\Omega_1) = (O\Omega_2) = sd(s^2 + R^2)^{-1/2}$$

and

$$(K\Omega_1) = (K\Omega_2) = Rd(s^2 + R^2)^{-1/2}.$$

In the case where $n=3$, that is, when the polygon is a triangle, the above formulas become

$$\rho^2 = \tfrac{1}{4}R^2 \sec^2 \omega = R^2(a^2b^2 + b^2c^2 + c^2a^2)/(a^2 + b^2 + c^2),$$
$$s = abc/(a^2 + b^2 + c^2), \qquad d^2 = R^2 - 3s^2.$$

These are the formulas which were given by Lemoine in 1873†.

COROLLARY. *The first Lemoine circle is the same for all the cyclic n-gons of the same kind inscribed in the same circle.*

8. Tucker circles. The first Lemoine circle of a cyclic n-gon may be considered as a particular case of the circles about to be mentioned and which may, from the analogous circles in the geometry of the triangle, be called Tucker circles.

THEOREM 4. *If two inscriptible cyclic n-gons of the same kind are directly homothetic, having the Lemoine point K for center of homothety, then each side of one cuts the sides of the other which are adjacent to the corresponding parallel side, in points on a circle.*

The proof of this theorem is analogous to that of Theorem 3.

THEOREM 5. (Second Lemoine circle) *If, through the Lemoine point K of an inscriptible cyclic n-gon* $A_1A_2A_3 \cdots A_n$, *we draw antiparallels to the diagonals* A_1A_3, A_2A_4, A_3A_5, $\cdots$, *the intersections of each of them with the sides adjacent to the corresponding diagonals lie on a common circle.*

This follows because each antiparallel has for length

$$2s = 2R \tan \omega = \text{constant},$$

and has K as midpoint.

COROLLARY. *The second Lemoine circle is the same for all the cyclic n-gons of the same kind inscribed in the same circle.*

9. Generalization of the preceding theorems. We conclude by generalizing some of the above theorems.

THEOREM 6. *The center and axis of collineation of any projectivity* π *on a circle are the Lemoine point and line of the figure formed by the chords which join, in pairs, the corresponding points of* π.

Let the two point sets $(A_1, A_2, A_3, A_4, \cdots)$ and $(A_1', A_2', A_3', A_4', \cdots)$

† Lemoine, Nouvelles Annales de Mathématiques, 1873, p. 364, and Association française pour l'avancement des sciences, 1873, p. 91, VI.

be projective on a circle, and let K and u be their center and axis of collineation.

Consider any two chords A_1A_1' and A_2A_2'. The lines A_1A_2' and $A_1'A_2$ intersect in a point M on the axis u. If A_{12} is the point of intersection of the chords A_1A_1' and A_2A_2', then for all points on the line MA_{12} the ratio of the distances to the two considered chords is equal to the ratio of the lengths of these two chords.

The polar of M with respect to the circle passes through the center of collineation K and through the point A_{12}. It is, moreover, the fourth harmonic of the two chords and the line MA_{12}. Consequently A_1A_2, KA_{12}, $A_2'A_1'$ are concurrent in a point N, and the ratio of the distances of any point on NA_{12}, and therefore of the point K, from the two chords, is equal to the ratio of the lengths of these same chords. Obviously one arrives at the same conclusion for all pairs of chords.

We see, then, that the point K is the Lemoine point of the figure formed by the chords A_1A_1', A_2A_2', $\cdots$. Similarly, the line u is the Lemoine line of the same figure.

COROLLARY. *The figure formed by the chords mentioned above possesses Brocard points, a Brocard circle, and first and second Lemoine circles.*

The Brocard points are the foci of the conic C which envelops the chords in question.

The projections of the Brocard points on the chords lie on the principal circle of the conic C.

As for the radii of the Lemoine circles, the distances of the Lemoine and Brocard points from the center of the circumcircle, and the value of tan ω, we have the same corresponding formulas.

Definitions

1. For a 6-gon with successive vertices A_1, A_2, A_3, A_4, A_5, A_6, we have a 6-gon of the first kind (the figure $A_1A_2A_3A_4A_5A_6$), a 6-gon of the second kind (the figure formed by the two triangles $A_1A_3A_5$, $A_2A_4A_6$), and a 6-gon of the third kind (the figure formed by the three double lines A_1A_4, A_2A_5, A_3A_6).

2. If, in the plane of a polygon, there exists a point K such that the distances of the point to the sides of the polygon are proportional to the lengths of these sides, this point is called the *Lemoine point* of the polygon. If, moreover, the polygon is inscribed in a circle, the polar of K with respect to the circle is called the *Lemoine line* of the polygon.

3. If, in the plane of a polygon, there exist two points Ω_1 and Ω_2 such that

$$\sphericalangle\, \Omega_1A_1A_2 = \sphericalangle\, \Omega_1A_2A_3 = \cdots = \sphericalangle\, \Omega_1A_nA_1$$
$$= \sphericalangle\, \Omega_2A_2A_1 = \sphericalangle\, \Omega_2A_3A_2 = \cdots = \sphericalangle\, \Omega_2A_1A_n = \omega,$$

these points are called *Brocard points* of the polygon.

4. A circle passing through two successive vertices of a polygon and tangent at one of these points to either the preceding or the following side of the polygon is called an *adjoint circle* of the polygon.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

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THE SIEVE OF ERATOSTHENES AND THE MÖBIUS STRIP

H. T. MCADAMS, Bethalto, Illinois

The Möbius strip is formed from a long rectangle of paper by rotating one end of the strip 180° relative to the other end and joining the two ends together. The resulting strip is a one-sided surface. It may likewise be shown that the strip will be one-sided for all rotations which represent odd multiples of 180°, but will be two-sided for all rotations which represent even multiples of 180°.

The ideal plane strip, prior to deformation into the Möbius figure, may be considered as a prism whose base has only two sides. From the practical standpoint, however, the strip has a finite thickness. This consideration forms the basis for a similar study of prisms of 3, 4, 5, $\cdots$, N sides when deformed in a manner analogous to that of the Möbius strip.

Suppose, for example, that the two bases of a regular triangular prism are joined after one base has been rotated about the long axis of the prism by 120° with respect to the other base. All three sides of the prism are now continuous. The same will be true for a similar rotation of 240°. However, for rotations of 0° and 360°, the figure will present three sides. These are the only possibilities, since they will be repeated upon further twisting, depending upon whether the total rotation of one base relative to the other is 360 N°, 360 N°+120°, or 360 N°+240°, where N is an integer.

In the case of the square prism, there may be 1, 2, or 4 "sides" depending upon the extent of the twist prior to the joining of the ends. If one base is rotated relative to the other base by 0° or by any multiple of 360°, four sides will of course be presented. If, however, the rotation is represented by 90° +360 N° or 270°+360 N°, there will be only one side. For all rotations represented by 180°+360 N° two sides will result. These possibilities will be repeated indefinitely with continued twisting of the prism prior to the joining of the ends.

By a generalization of this process, the possibilities may be derived for any prism of number of sides S when one base is rotated by multiples of 360°/S, relative to the other base. The following array gives the number of surfaces obtained for prisms up to ten sides, when twisted by $0\cdot(360°/S)$, $1\cdot(360°/S)$, $2\cdot(360°/S)$, $\cdots$, $S(360°/S)$. The first number in each horizontal represents the number of sides of the regular prism at 0° rotation. Successive numbers from left to right represent the number of surfaces obtained for the first, second, etc. multiples of 360°/S.

It is apparent that every horizontal repeats itself both diagonally and vertically. It is also obvious that each row of the array may be generated from purely numerical considerations by setting down, from left to right, the highest factor

common to the number of sides represented by that row and the successive integers 1, 2, 3, · · · , and so on. For example, in the 6-horizontal, the numbers 1, 2, 3, 2, 1 are the highest common factors respectively of 6 and 1, 6 and 2, 6 and 3, 6 and 4, and 6 and 5.

			2	1	2					
			3	1	1	3				
		4	1	2	1	4				
		5	1	1	1	1	5			
	6	1	2	3	2	1	6			
	7	1	1	1	1	1	1	7		
8	1	2	1	4	1	2	1	8		
9	1	1	3	1	1	3	1	1	9	
10	1	2	1	2	5	2	1	2	1	10

From the above considerations it is to be noted that the only integer occurring in a row representing a prime number, other than itself, is 1. The requisite condition for the occurrence of a prime horizontal, therefore, is the alignment of 1's in all prime diagonals. This eliminates every second number after 2, every third number after 3, every fifth number after 5, and so on, thereby duplicating the Sieve of Eratosthenes.

A NON-SINGULAR POLYHEDRAL MÖBIUS BAND WHOSE BOUNDARY IS A TRIANGLE

BRYANT TUCKERMAN, Cornell University

A Möbius band is a non-orientable surface which can be defined abstractly by proper identification of the opposite ends of a rectangle, with a sense opposite to that which would make the rectangle into a cylinder. Its boundary is a single simple closed curve.

A physical model of a Möbius band is customarily made by pasting together the ends of a long narrow strip of paper after twisting one end through 180° with respect to the other. The boundary of the usual model is an unknotted closed space curve. Another model of the Möbius band in three-dimensional space is the "cross-cap," of which the boundary may be a circle. However, the cross-cap has a line of self-intersections, that is, it is singular.

It does not seem well-known that there are non-singular models (that is, homeomorphs) of the Möbius band in three-dimensional Euclidean space, such that the boundary is a plane curve, for example a circle. However, such models are discussed and illustrated in *Anschauliche Geometrie*, by D. Hilbert and S. Cohn-Vossen (Berlin, Springer, 1932; New York, Dover, 1944), pp. 278–279, figs. 307–308. (The cross-cap is shown in fig. 310, and the usual model of the Möbius band is on p. 269.) Those models have curved surfaces.

The purpose of this note is to present a convenient *polyhedral*, non-singular model of a Möbius band, in three-dimensional Euclidean space, whose boundary lies in a plane and is in fact a triangle. Its seven vertices are the six vertices and the centroid of a regular octohedron. Its nine faces are six of the equilateral triangular faces of the octohedron, and one large and two small isosceles right triangular faces each containing the centroid of the octohedron. The boundary is a triangle consisting of two edges and a diagonal of the octohedron. (The triangle could be deformed to a circle by an isotopic, but naturally not polyhedral, deformation of the space.) By bisection of the large right triangle, the model can be made into a simplicial complex having ten 2-cells, seventeen 1-cells, and seven 0-cells.

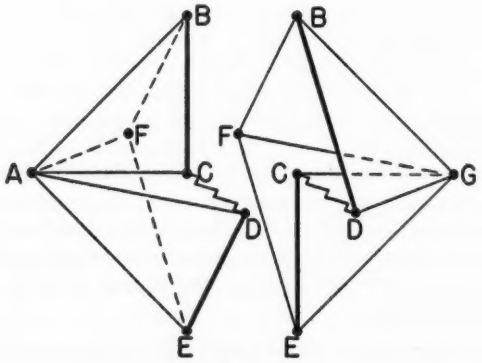

FIG. 1

Figure 1 shows the model after being cut into two pieces which were then pulled apart. Along the back the cut is along two edges of the octohedron; in the front the cut (shown jagged) is through the large right triangle. This decomposition makes it evident that the model represents a Möbius band. The triangular boundary is indicated by heavy lines. It will be noted that the Möbius band meets the plane of its boundary both inside and outside the boundary.

This model is related to the "heptahedral" model of the projective plane (ibid., pp. 266 ff.). Both are subsets of the figure whose faces are the faces of a regular octohedron together with three square faces whose vertices are vertices of the octohedron. But this model of the Möbius band is not a subset of the heptahedron, nor is it singular as is the heptahedron.

The model is readily constructed from equilateral and isosceles right triangular faces joined as in Figure 2. The jagged line corresponds to the jagged cut in Figure 1, and can be omitted. The heavy lines will be the boundary of the model. In each triangular face, the + or − sign at an edge represents the direction of the bend at that edge in the finished model, as viewed from the side of the face which is on top in Figure 2. To construct the model, bend along the interior

edges, downward (say) where marked +, upward where marked −, and juxtapose and seal together the two edges of each of the four pairs marked AC, GC, BF, EF.

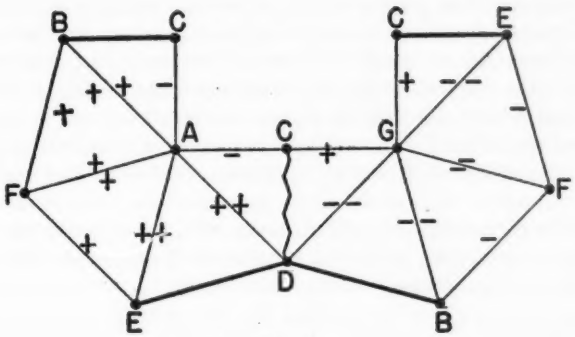

FIG. 2

Two modifications of this model, having fewer faces, may be mentioned. If, in Figure 1, the vertices B and E are displaced toward the rear, then the vertex F and the four triangular faces which meet at F may be replaced by two triangular faces ABE and GBE. There results a model having seven triangular faces. On the other hand, if the vertex C is displaced toward the front, then the vertex F and the four triangular faces which meet at F may be replaced by a square face $ABGE$, while the triangular face AGD becomes a reëntrant quadrilateral face $ACGD$. There results a model having six faces, namely four triangular faces, a square face, and a reëntrant quadrilateral face. In both of these modifications, the boundary of the Möbius band is a reëntrant plane quadrilateral.

It is a pleasure to acknowledge my indebtedness to Professor H. Hopf for interesting conversations with him while he was in this country a year ago, since the idea for this note arose in one of these discussions.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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ONE-SIDED MAXIMA AND MINIMA

J. D. MANCILL, University of Alabama

1. Introduction. Repeated inquiries from students and colleagues concerning the subject of this discussion, doubtless due to the incomplete treatment of it in the literature, have prompted this attempt to give a complete analysis of the

problem of one-sided (unilateral) extremes for functions of a single variable. Almost without exception, the text books on the calculus restrict their discussion of maxima and minima of functions to the interior points of the range of definition of the function. The author knows of no text which treats fully sufficient conditions for an end-point of the range to be a maximum or a minimum. When the end-point extremes are mentioned they are frequently referred to as absolute extremes, since the usual definition of relative extreme excludes the end-points. But this is undesirable since the end-point extreme may not be absolute. It is better to state the definition of relative extreme so as to include the end-points, and develop necessary conditions and sufficient conditions that an end-point be a relative maximum or minimum. Then the procedure, even in elementary calculus, would be to test the end-points along with the zeros of the first derivative for relative extremes, and then determine the absolute extremes of the function from all the extremes on the closed range.

In a recent note Oakley* notes that the end-point extremes may be transformed into interior point extremes by a suitable transformation on the independent variable. Although this explains why a point may be an end-point maximum in one formulation of a problem and an interior maximum in another, such a transformation is not a practical way of handling end-point maxima. For first, it renders even the simple problems very complicated, especially when it comes to checking the sufficient conditions for a maximum or a minimum. Second, it is entirely unnecessary. This is due to the fact that an end-point is in general an extreme and one only needs to determine whether it is a relative maximum or a relative minimum. Some examples will illustrate these remarks.

2. **Functions of one variable.** We shall consider a single-valued function

$$y = f(x), \qquad a \leqq x \leqq b,$$

defined on the closed interval (a, b). The following definitions will be needed:

DEFINITION 1. $f(x)$ belongs to the class C of continuous functions at $x=x_1$, $a<x_1<b$ if and only if $\lim_{x\to x_1} f(x)=f(x_1)$.

DEFINITION 2. $f(x)$ belongs to the class of functions C at the end-point a if and only if $\lim_{x\to a+} f(x)=f(a)$ and at the end-point b if and only if $\lim_{x\to b-} f(x)=f(b)$.

DEFINITION 3. $f(x)$ belongs to the class of functions C^n if and only if $f^{(n)}(x)$ belongs to C.

DEFINITION 4. $f'_+(x_1)=\lim_{h\to 0+}[f(x_1+h)-f(x_1)]/h$, and similarly for $f'_-(x_1)$.

DEFINITION 5. $f(x)$ belongs to the class of functions D' on (a, b) if and only if $f(x)$ belongs to C on (a, b) and $f'(x)$ belongs to C on (a, b) except at a finite number of points x_1 at which the derivatives $f'(x_1+)$ and $f'(x_1-)$, on the right and left of x_1 respectively, exist, where $f'(x_1+)=\lim_{x\to x_1^+} f'(x)$ and similarly for $f'(x_1-)$.

DEFINITION 6. $f(x)$ has an *absolute maximum* at a point x_1 of the interval (a, b) if and only if $f(x_1)\geq f(x)$ for all values of x in (a, b). A function $f(x)$ has a

* End-point maxima and minima, this MONTHLY, vol. 54 (1947), pp. 407–409.

relative maximum at a point x_1 of (a, b) if and only if $f(x_1) \geqq f(x)$ for all values of x on (a, b) in a certain neighborhood of x_1.

Obvious modifications of the inequalities are necessary for the definition of absolute and relative minima. A well-known property of continuous functions states that a function $f(x)$ which is continuous on the closed interval (a, b) takes its absolute maximum and minimum values at least once in that interval. Therefore, one must consider the end-points of the closed interval in determining the absolute extremes of the function on that interval.

The reader should distinguish between $f'_+(x_1)$ and $f'(x_1+)$. This is easily done by considering the function $x^2 \sin(1/x)$. In this case $f'_+(0)=0$ but $f'(0+)$ does not exist.

The geometric meaning of Definition 5 is that the curve defined by the function $y=f(x)$ on the range (a, b) has a finite number of corners, or in other words, the curve is composed of a finite number of sub-arcs each of class C'.

THEOREM 1. *If a continuous function $f(x)$ assumes a relative maximum value at $x=x_1$ of the interval (a, b); and if $f'_-(x_1)$ exists, then $f'_-(x_1) \geqq 0$; and if $f'_+(x_1)$ exists, then $f'_+(x_1) \leqq 0$. The inequalities are reversed if x_1 is a minimum.*

This theorem follows immediately from Definition 3 and the inequalities

$$[f(x_1 + h) - f(x_1)]/h \geqq 0 \qquad \text{for } h < 0,$$

$$[f(x_1 + h) - f(x_1)]/h \leqq 0 \qquad \text{for } h > 0,$$

for the case when x_1 is a maximum.

COROLLARY (FERMAT'S THEOREM). *If a function $f(x)$ assumes a relative maximum or a minimum at $x=x_1$, $a<x_1<b$ at which $f(x)$ is differentiable, then $f'(x_1)=0$.*

The corollary follows from the fact that $f'(x_1)$ exists if and only if $f'_-(x_1)$ and $f'_+(x_1)$ exist and are equal. The end-points are excluded from the corollary since the left-hand derivative at $x=a$ and the right-hand derivative at $x=b$ are not considered in the theorem.

THEOREM 2. *A continuous function $f(x)$ assumes a relative maximum at $x=x_1$; x_1 interior to (a, b) if $f'_-(x_1)>0$ and $f'_+(x_1)<0$; at $x=a$ if $f'_+(a)<0$; and at $x=b$ if $f'_-(b)>0$. Similar conditions with inequalities reversed are sufficient for a minimum.*

These conclusions follow immediately from Definition 4. The function $f(x) = 1-|x|$, $-1 \leqq x \leqq 1$ affords a simple illustration of these results. In this example $f'_-(0)=1$ and $f'_+(0)=-1$. Therefore, $f(x)$ has a relative maximum at $x=0$ which is also the absolute maximum on that range.

THEOREM 3. *If $f(x)$ is of class C^n within (a, b), $f^{(n)}(a+)$ exists, and*

$$f^{(k)}(a) = 0, \qquad k = 1, \cdots, n-1,$$

$$f^{(n)}(a+) < 0,$$

then $f(x)$ has a relative maximum at $x=a$. Similar conditions with the inequalities

reversed are sufficient for a minimum.

From Taylor's formula and the hypotheses of the Theorem, we have

$$f(x) - f(a) = (x - a)^n f^{(n)}(t)/n!, \qquad a < t < x.$$

The sign of the right member of this equation depends only upon the sign of $f^{(n)}(t)$, since $x > a$. Also, t approaches a with x and therefore, if x is sufficiently near a the sign of the right member is negative.

As an example consider the function $f(x) = -x^{3/2}$, $0 \leqq x \leqq 1$. It is easily seen that $f'(0) = 0$ and $f''(0+) = -\infty$. Therefore $f(x)$ has a relative maximum at $x = 0$ which is also the absolute maximum on that range.

THEOREM 4. *If $f(x)$ is of class C^n within (a, b), $f^{(n)}(b-)$ exists and*

$$f^{(k)}(b) = 0, \qquad k = 1, \cdots, n - 1,$$
$$f^{(n)}(b -) < 0, \qquad n \text{ even},$$
$$f^{(n)}(b -) > 0, \qquad n \text{ odd},$$

then $f(x)$ has a relative maximum at $x = b$. Similar conditions with the inequalities reversed are sufficient for a minimum.

It follows from the hypotheses of the theorem and from Taylor's formula that

$$f(x) - f(b) = (x - b)^n f^{(n)}(t)/n!, \qquad x < t < b.$$

In this case the sign of the right member of this equation depends upon the sign of $f^{(n)}(t)$ and whether n is even or odd since $x < b$. Therefore, the conclusions of the theorem follow as in the proof of Theorem 3.

As an example consider the function $f(x) = -(-x)^{5/2}$, $-1 \leqq x \leqq 0$. It is easily seen that $f'(0) = f''(0) = 0$, and $f^{(3)}(0-) = +\infty$. Therefore, $f(x)$ has a relative maximum at $x = 0$ which is also the absolute maximum on that range.

We may say that $f(x)$ on (a, b) has a *left-hand maximum* at $x = x_1$ if $f(x_1) \geqq f(x)$ for all values of x in a certain right hand neighborhood of x_1, for example in Theorem 3. Likewise, we may say $f(x)$ has a *right-hand maximum* at $x = x_1$ if $f(x_1) \geqq f(x)$ for all values of x in a certain left-hand neighborhood of x_1, for example in Theorem 4. Obviously then, if *x_1 is interior to the interval (a, b) and $f(x)$ has a left-hand and a right-hand maximum at x_1, at which $f(x)$ is continuous, then $f(x)$ has a maximum at $x = x_1$.* For example, the function

$$f(x) = -(-x)^{5/2}, \qquad -1 \leqq x \leqq 0,$$
$$= -x^{3/2}, \qquad 0 \leqq x \leqq 1,$$

has a maximum at $x = 0$. The following well-known theorem follows easily:

THEOREM 5. *If $f(x)$ is of class C^{2n} on (a, b), x_1 is interior to (a, b), and*

$$f^{(k)}(x_1) = 0, \qquad k = 1, \cdots, 2n - 1,$$
$$f^{(2n)}(x_1) < 0,$$

then $f(x)$ has a maximum at $x=x_1$. Similar conditions with inequalities reversed are sufficient for a minimum.

It may also be remarked here that if a continuous function $f(x)$ has a left-hand maximum and a right-hand minimum (or vice-versa) at x_1 interior to (a, b), then $f(x)$ has an inflection at x_1. If $f'_-(x_1) \neq f'_+(x_1)$, then $f(x)$ has an abrupt inflection at x_1. For example the function

$$f(x) = -(-x)^{5/2}, \qquad -1 \leqq x \leqq 0,$$
$$= x^{3/2}, \qquad 0 \leqq x \leqq 1,$$

has an inflection at $x=0$.

3. **An application.** As a physical application, let us consider one of the problems treated by Oakley in his note already referred to:

A man is in a boat at P one mile from the nearest point A on shore. He wishes to go to B which is farther down the shore M miles from A. If he can row r miles an hour and walk w miles an hour, toward what point C should he row in order to reach B in least time?

Assume the shore line to be straight with the point B to the right of A and let $x=AC$. Then the time $t(x)$ that it takes to go from P to B is

$$t(x) = (1+x^2)^{1/2}/r + (M-x)/w, \qquad 0 \leqq x \leqq M,$$

whence

$$t'(x) = x/r(1+x^2)^{1/2} - 1/w,$$

and $t'(0)<0$. Therefore, $t(x)$ has a relative maximum at $x=0$ for all positive values of r, w, and M, and consequently the man would not row towards A. If $r \geqq w$, then $t'(M)<0$ and $t(x)$ has a relative minimum at $x=M$. This is the absolute minimum since in this case $t'(x) \neq 0$ on $0 \leqq x \leqq M$. Therefore, the man should row directly to B. If $r<w$, then $t'(w)=0$ has the unique solution

$$x_C = r/(w^2-r^2)^{1/2}.$$

It is easily shown that in this case $t(x_C)<t(M)$ if $0<x_C<M$, and consequently the man should row to x_C.

There is still one other case which was not considered by Oakley, that is when $x_C=M$, or when $r=wM/(1+M^2)^{1/2}<w$. In this case $t'(M)=0$ and $t''(M)>0$. It follows from Theorem 4 that $t(x)$ has a relative minimum at $x=M$ and this is the absolute minimum since $t'(x)$ vanishes only at x_C. Therefore, the man should row directly to B in this case.

It is the author's firm conviction that if maxima and minima problems, even in elementary calculus, are analyzed as exemplified in these pages, the student will meet no difficult or astonishing "answers." This is due in part to the fact that the student will not be led to the erroneous conclusion that maxima and minima occur only at the vanishing of the first derivative.

FERMAT'S PRINCIPLE AND CERTAIN MINIMUM PROBLEMS

A. V. BAEZ, Stanford University

The object of this note is to show the relation between Fermat's Principle and the end-point minimum problems discussed by Oakley.*

It is known that of all possible paths connecting two points A and B in space a light ray leaving A chooses the path which enables it to reach B in the least time. This is Fermat's Principle from which the laws of reflection and refraction follow. They can be used to solve certain minimum problems without differentiation. Before discussing Oakley's problems consider a simpler one.

A farmer has to walk from his house to a straight river, fill his pail and carry it to his barn in the least time. At what point should he fill his pail? The solution by geometrical optics is obvious. His path to and from the river must make equal angles with the normal to the river.

Suppose he always walks to the river at a speed v_1 and away from it at a speed v_2. Where then should he dip his pail? Snell's Law immediately suggests that his path of approach to the river should make an angle θ_1 and that away from the river an angle θ_2 with the normal so that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

The problem which follows was discussed by Oakley, and his notation is used, together with the above equation which expresses Snell's Law.

PROBLEM. *A man is in a boat at P one mile from the nearest point A on shore. He wishes to go to B which is farther down the shore M miles from A. If he can row r miles an hour and walk w miles an hour, toward what point C should he row in order to reach B in least time?*

Let $AC=x$. Snell's Law indicates that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{r}{w}.$$

Use $\theta_2=90°$ and geometry to obtain

$$\frac{x}{\sqrt{1+x^2}} = \frac{r}{w}.$$

Solving for x we have

$$x = \frac{r}{\sqrt{w^2-r^2}}$$

which Oakley obtains by calculus. The optical analysis makes clear why the result is independent of M(θ_1 is the "critical" angle). Optically it is also apparent that if $r<w$ and $r\geq M\sqrt{w^2-r^2}$ or if $r\geq w$ (giving $\sin\theta_1\geq 1$) the man should go straight to B.

* End-Point Maxima and Minima, this MONTHLY, vol. 54 (1947) pp. 407–409.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 816. *Proposed by P. L. Chessin, New York, N. Y.*

A announces a two digit number from 01 to 99. B reverses the digits of this number and adds to it the sum of its digits and then announces *his* result. A continues in the same pattern. All numbers are reduced modulo 100, so that only two digit numbers are announced. What choices has A for the initial number in order to insure that B will at some time announce 00?

E 817. *Proposed by E. V. Hofler, Colgate University*

If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

E 818. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find four different fractions, each of the form $m/(m+1)$, such that their sum is an integer. (For example: $1/2+2/3+6/7+41/42=3=1/2+2/3+8/9+17/18$.)

E 819. *Proposed by H. F. Sandham, Trinity College, Ireland*

If $$S_n = 1/1 + 1/2 + \cdots + 1/n,$$

prove that

$$\gamma < S_p + S_q - S_{pq} \leqq 1,$$

where γ is Euler's constant.

E 820. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If ABC is an equilateral triangle, and P is any point on the circumference of the inscribed circle, prove synthetically that $(PA)^2+(PB)^2+(PC)^2$ is constant.

SOLUTIONS

Eadem Numero Mutata Resurgo

E 784 [1947, 412]. *Proposed by R. E. Gaines, University of Richmond*

Show that the locus of the intersection of two successive perpendicular

tangents to a logarithmic spiral is another logarithmic spiral.

Solution by Ernest Trost, Zürich, Switzerland. We shall establish a more general result. Let $\alpha \leqq 90°$ be the constant angle between a tangent t to the logarithmic spiral and the radius $r = A^\theta$ drawn through the point of contact, $P(r, \theta)$. If $\beta \leqq 90°$ be the *constant* angle between t and another tangent t_1 at $P_1(r_1, \theta_1)$, then $\theta_1 - \theta = \beta$, and the angles β, $180° - \alpha$, $180° - \beta$, α of the quadrangle Q with sides r, t, t_1, r_1 are constant. Since $r_1 = rA^\beta$, Q is unaltered in form as P moves on the spiral L. Thus the locus of each point of Q is a spiral congruent with L after a suitable turning about the pole.

Also solved, by analytical methods, by Ragnar Dybvik, R. T. Hood, Roscoe Woods, and the proposer. The above solution appears in a footnote of a paper by Trost, *Zur Theories der isoptischen Kurven*, to appear in *Nieuwe Archief voor Wiskunde.*

Equilateral Triangle About a Regular Polygon

E786 [1947, 471]. *Proposed by Michael Goldberg, Washington, D.C.*

Suppose that an equilateral triangle is circumscribed about a regular n-gon, where $n = 3k \pm 1$, so that one side of the n-gon lies on one of the sides of the triangle. Show that the angle subtended by this side of the n-gon at the opposite vertex of the triangle is $2\pi/3n$.

Solution by the Proposer. Let D be a vertex of the given n-gon touched by a side of the circumscribing equilateral triangle. Let P be the foot of the perpendicular from the center O of the n-gon upon the base AB, where A is chosen on the same or opposite side of PO according as $n = 3k-1$ or $3k+1$. Let the angle AOP be 3θ. Locate C on PO, and the point E, so that $EO = EC = OA$ and the angles EOC and ECO are equal to θ. Then the points C, E, A are collinear. Join C to D. Then

$$\begin{aligned} \text{angle } DOE &= \text{angle } DOC \pm \theta \\ &= k\pi/n \pm \pi/3n = \pi(3k \pm 1)/3n = \pi/3, \end{aligned}$$

since $3k \pm 1 = n$. Therefore triangle DOE is equilateral, and it follows in all cases that angle DCO is $\pi/6$, and that C is the vertex of the circumscribing equilateral triangle.

Also solved by Roger Lessard.

Editorial Note. The above solution proves that if regular polygons having $3k \pm 1 (k = 1, 2, \cdots)$ sides be inscribed in the same circle $O(A)$, and all have a common vertex A, then the equilateral triangles circumscribed about these polygons and having one side coincident with a side of the polygon issuing from A, have their opposite vertices on the trisectrix with center O and double point A.

Elaboration of Heron's Formula

E 787 [1947, 471]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, show that

$$s^4 + (s-a)^4 + (s-b)^4 + (s-c)^4 - a^4 - b^4 - c^4 = 12S^2,$$

where a, b, c are the sides, s the semi-perimeter, and S the area.

I. *Solution by Leo Moser, University of Manitoba.* Replacing s and S by their known values in terms of a, b, c, it becomes necessary and sufficient to show that

$$\left(\frac{a+b+c}{2}\right)^4+\left(\frac{-a+b+c}{2}\right)^4+\left(\frac{a-b+c}{2}\right)^4 + \left(\frac{a+b-c}{2}\right)^4 - a^4 - b^4 - c^4 = 12\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right).$$

If we consider the left side as a polynomial of degree four in a, and note that it vanishes for $a=b+c$, $a=b-c$, $a=-b+c$, $a=-b-c$, then we get the identity apart from the numerical factor 12/16 on the right side. This may be determined by taking, say, $a=1$, $b=c=0$.

The result is essentially the same as that given (with proof) in art. 808, p. 534, of Todhunter's *Algebra*, 1885 edition.

II. *Solution by M. LeLeiko, Rutgers University.* In the identity

$$(p+q+r)^4 + p^4 + q^4 + r^4 = (p+q)^4 + (q+r)^4 + (r+p)^4 + 12pqr(p+q+r),$$

set $p=s-a$, $q=s-b$, $r=s-c$. Then $p+q+r=s$, $p+q=c$, $q+r=a$, $r+p=b$. With the aid of Heron's formula the result now follows.

Similar results may be obtained by making the same substitutions in the identities

$$(p+q+r)^3 + p^3 + q^3 + r^3 = (p+q)^3 + (q+r)^3 + (r+p)^3 + 6pqr$$

and

$$(p+q+r)^2 + p^2 + q^2 + r^2 = (p+q)^2 + (q+r)^2 + (r+p)^2.$$

Also solved by Daniel Block, W. E. Buker, P. L. Chessin, Monte Dernham, Ragnar Dybvik, J. C. Eaves, J. F. Heyda, Meyer Karlin, B. R. Leeds, Roger Lessard, Eleanor Ranking, O. M. Rasmussen, Kaidy Tan, C. W. Trigg, and the proposer.

Several solvers made the observation that a, b, c may be *any* numbers (not only those capable of representing the sides of a triangle) if we replace s by $(a+b+c)/2$ and S^2 by $s(s-a)(s-b)(s-c)$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4295. *Proposed by Irving Kaplansky, University of Chicago*

Show that any group with more than two elements admits an automorphism other than the identity.

4296. *Proposed by H. S. Wall, University of Texas*

It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \cdot \\ \quad \cdot \\ \quad \cdot}}$$

converges, then its value is the numerically larger root of the equation $x^2-px+q=0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

4297. *Proposed by Paul Erdös, Syracuse University*

Put $\phi_k^{(n)} = \sum r_1 r_2 \cdots r_k$, where the r's run through all integers $\leqq n$, and the r's are all different. $\phi_n^{(n)} = n!$ It is well known that $\phi_{n-1}^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$. Prove that if n is sufficiently large $\phi_k^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$ for any k. (This is not always true; e.g. $n=3$, $k=1$.)

4298. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, the incenter of which is I, the perpendicular at A to the faces ACD, ADB, ABC respectively cut the planes ICD, IDB, IBC in A_1, A_2, A_3. (1) The perpendicular through A to the plane $A_1A_2A_3$ passes through the point of contact of the inscribed sphere with the opposite face BCD. (2) The analogous property is true for a triangle ABC.

4299. *Proposed by R. C. Lyness, Crakemarsh, Uttoxeter, Staffordshire, England*

If the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares; e.g., $8^3-7^3=(3^2+2^2)^2$.

SOLUTIONS

Six Spheres Associated with a Tetrahedron

3981. [1941, 69] *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let S_i, $(i=1, 2, 3, 4, 5, 6)$, be the spheres of similitude of the spheres with centers at the vertices of the tetrahedron $A_1A_2A_3A_4$ such that the square of the radius of any one of the latter is equal to one-half of the sum of the squares of the edges of the opposite face. (1) Examine the relative positions of the spheres S_i. (2) Show that these six spheres are orthogonal to the circumspheres of $A_1A_2A_3A_4$ and of its anticomplementary tetrahedron. (3) The powers of the extremities of an edge of $A_1A_2A_3A_4$ with respect to the sphere S_i whose center is on the opposite edge are independent of the length of that last edge. (4). The spheres S_i' symmetric to the spheres S_i with respect to the midpoint of the edges on which they are centered intersect in two points collinear with the circumcenter of $A_1A_2A_3A_4$.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We shall designate the sphere (S_j) whose center ω_{ik} is on the edge A_iA_k by (ω_{ik}). Let $A_2A_3=a$, $A_1A_4=a'$, $A_1A_3=b$, $A_2A_4=b'$, $A_1A_2=c$, $A_3A_4=c'$, and let h_i be the length of the altitude drawn from A_i. If the tetrahedron $A_1A_2A_3A_4$ is taken as the reference tetrahedron, the equation in normal tetrahedral coördinates of the circumsphere (0) of $A_1A_2A_3A_4$ may be written as

$$S \equiv \sum \frac{a^2yz}{h_2h_3} = 0.$$

Let R_i denote the radius of the given sphere of center A_i.

(1) The six points ω_{ik} are coplanar. We have

$$\overrightarrow{A_1\omega_{12}} = \frac{R_1^2}{R_1^2-R_2^2}\overrightarrow{A_1A_2}, \qquad \overrightarrow{A_1\omega_{13}} = \frac{R_1^2}{R_1^2-R_3^2}\overrightarrow{A_1A_3},$$

$$\overrightarrow{A_1\omega_{14}} = \frac{R_1^2}{R_1^2-R_4^2}\overrightarrow{A_1A_4}.$$

The plane (P) determined by ω_{12}, ω_{13}, ω_{14} has as its equation

$$P \equiv \sum \frac{R_1^2x}{h_1} = 0$$

which is satisfied by the coördinates of all six points ω_{ik}. It may be noted that (P)

* Translated and corrected by W. E. Byrne, Virginia Military Institute.

is the radical plane of the sphere (O) and the sphere orthogonal to the given spheres (A_i).*

(2) The sphere (ω_{12}) is the locus of points M such that

$$\frac{MA_1}{R_1} = \frac{MA_2}{R_2}.$$

Since the extremities of its diameter situated on A_1A_2 are harmonic conjugates with respect to A_1 and A_2, (ω_{12}) is orthogonal to (O). Therefore all the spheres (ω_{ik}) have the same radical axis, the perpendicular to (P) through O. Since

$$\overline{A_1M}^2 = h_1S_x' - S,$$
$$\overline{A_2M}^2 = h_2S_y' - S,$$

the equation of (ω_{12}) is:

$$(R_1^2 - R_2^2)S + (h_1R_2^2S_x' - h_2R_1^2S_y')\sum (x/h_1) = 0.$$

The absolute normal coördinates of the centroid G are $(\frac{1}{4}h_1, \frac{1}{4}h_2, \frac{1}{4}h_3, \frac{1}{4}h_4)$. The power $\tilde{\omega}(G)$ of G with respect to (ω_{12}) is given by:

$$-\tilde{\omega}(G) = S(\tfrac{1}{4}h_1, \tfrac{1}{4}h_2, \tfrac{1}{4}h_3, \tfrac{1}{4}h_4) + \frac{1}{R_1^2 - R_2^2}\left[\tfrac{1}{4}R_2^2(\sum a^2 - 2R_1^2) - \tfrac{1}{4}R_1^2(\sum a^2 - 2R_2^2)\right],$$

which reduces to

$$\tilde{\omega}(G) = \frac{3}{16}\sum a^2.$$

If (O') is the circumsphere of the anticomplimentary tetrahedron† of tetrahedron $A_1A_2A_3A_4$, we have

$$\overrightarrow{GO'} = -3\overrightarrow{GO}.$$

If $\tilde{\omega}(O)$ and $\tilde{\omega}(O')$ are the power of O and O' with respect to (ω_{12}), then

$$\tilde{\omega}(O') + 3\tilde{\omega}(O) = 12\overline{OG}^2 + 4\tilde{\omega}(G).$$

But $OG^2 = R^2 - \sum a^2/16$, and $\tilde{\omega}(O) = R^2$, so that $\tilde{\omega}(O') = 9R^2$. Hence (O') is orthogonal to (ω_{12}).

(3) If $\tilde{\omega}(A_3)$ is the power of A_3 with respect to (ω_{12}):

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* The plane (P) is called the Newtonian plane of the spheres (A_i). Court, Modern Pure Solid Geometry, p. 203.

† Anticomplimentation with respect to $A_1A_2A_3A_4$ is defined by $\overrightarrow{GM'} = -3\overrightarrow{GM}$. Hence the circumradius R' of the anticomplimentary tetrahedron is $3R$, R being the circumradius of $A_1A_2A_3A_4$.

$$\bar{\omega}(A_3) = -\frac{1}{R^2 - R_2^2}[b^2R_2^2 - a^2R_1^2]$$
$$= \frac{b^2(a'^2 + b^2 + c'^2) - a^2(a^2 + b'^2 + c'^2)}{a'^2 + b^2 - (a^2 + b'^2)}$$

which does not depend on c.

(4) Let (ω'_{ik}) be symmetrical to (ω_{ik}) with respect to the midpoint of the edge A_iA_k. The centers ω'_{ik} are located in a plane (P'), the reciprocal transversal plane of (P) with respect to the tetrahedron $A_1A_2A_3A_4$.‡ The equation of (P') is:

$$P' \equiv \sum \frac{x}{n_1R_1^2} = 0.$$

The sphere (ω'_{ik}) is orthogonal to (O), since the extremities of the diameter of (ω'_{ik}) on A_iA_k are harmonic conjugates with respect to A_i and A_k. All the spheres (ω'_{ik}) have two points M_1, M_2 in common, situated on the diameter of (O) perpendicular to (P'). Both M_1 and M_2 satisfy the relation

$$\overline{MA_1} \cdot R_1 = \overline{MA_2} \cdot R_2 = \overline{MA_3} \cdot R_3 = \overline{MA_4} \cdot R_4.$$

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

An Introduction to Analytical Geometry. Vol. II. By A. Robson. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+215 pages. $2.50.

This book is the second volume of a work on Analytic Geometry. Since frequent reference is made to volume I and it is written in a terse style, it would not by itself prove suitable as a text for a course in geometry in most American schools. However, it could be a valuable reference to teachers of courses in advanced analytic, analytic projective, or synthetic projective geometry.

The contents of this book include material that is frequently covered in courses in real, complex, synthetic, analytic, metric, or projective geometry, a much wider range of topics than is dealt with in many courses in geometry. There are ten chapters, the first of which is numbered seventeen, entitled Homography, and deals with projectivities and perspectivities, including pro-

‡ Court, ibid., p. 122.

jectivities on a conic. Since theories of cross ratio, pole and polar, and parametric representation are developed in volume I, free use is made of these concepts. Chapter eighteen (the second in this volume) deals mainly with the theory of involutions from the algebraic point of view, and includes applications to conic sections. Proofs of Desargues' and Brianchon's theorems which utilize the theory of involutions are also presented. The next chapter is entitled General Geometry and contains a brief discussion of the different types of geometry and the coordinates used in each. Miscellaneous properties of conic sections are included in this chapter. Ranges on a Conic is the title of chapter twenty, which is a continuation of the discussion of projectivities on a conic begun in chapter seventeen. The twenty-first chapter deals with systems of conics, their classification and some of the properties of systems of conics. Chapter twenty-two is entitled Reciprocation and primarily deals with reciprocation with respect to a general conic (duality), reciprocation with respect to a circle being considered a special case of the general theory. The problem of the classification of curves represented by equations of the second degree is dealt with in chapter twenty-three. Chapter twenty-four is entitled Foci and Confocals, and includes a general definition of foci as well as a brief discussion of the properties of confocal conics. The last two chapters are entitled Normals and Evolutes, and Special Homogeneous Coordinates and deal with these topics, especial emphasis being placed on areal coördinates.

The virtues of this book include (1) a large number of well chosen illustrative examples, (2) a long list of exercises for the student, (3) examples showing how well known theorems can be proved by several different modes of procedure, and (4) examples illustrating how methods customarily associated with one type of geometry can be utilized to advantage in some other type of geometry. Its faults include (1) so many references to volume I that it can not be read satisfactorily by those who are not conversant with that work, (2) a style more terse than many American readers are accustomed to, and (3) (which some may consider a virtue), a vagueness in hypotheses regarding the type of geometry (metric, projective, real, complex, etc.) in which a given proof of a theorem is valid.

R. G. SANGER

NEW BOOKS RECEIVED

Les Principes Mathématiques de la Mécanique Classique. By M. Brelot. Grenoble, Arthaud, 1945. 62 pages. 120 fr.

Die Bewegungsgruppen der Kristallographie. By J. J. Burckhardt. Basel, Birkhauser, 1947. 186 pages. 29 fr.

Analytic Mechanics. By A. E. Currier. Annapolis, U. S. Naval Institute, 1948. 10+306 pages. $4.75.

Elements of Nomography. By R. D. Douglass and D. P. Adams. New York, McGraw-Hill Book Co., 1947. 9+209 pages. $3.50.

Calculus and its Applications. By R. D. Douglass and S. D. Zeldin. New York, Prentice-Hall, Inc., 1947. 8+568 pages. $5.15.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE. Club reports have been received in almost overwhelming numbers and it has been impossible thus far to include all of them on these pages. If the report of your chapter has not as yet appeared, it is probably due to the fact that your Editor has used the reports in the order in which he has received them.

In order to record the activities of your chapter for the year 1947–48, the club reports should be sent in immediately. When writing the report please follow the form which is used on these pages. The material will be included for publication in the order in which it is received, provided the reports are neatly written in the form prescribed.

CLUB REPORTS, 1946–47

Pi Mu Epsilon, University of Pennsylvania

The Pennsylvania Alpha Chapter of Pi Mu Epsilon, inactive during the war, was reorganized at the close of the fall semester of 1946–47. At that time sixteen undergraduates were elected to membership. Interest is growing rapidly among mathematics and science students on campus. During the year the following talks were given by members of the faculty and students:

Algebraic geometry, by Dr. I. S. Cohen

Polyhedra and the four-color problem, by Dr. Hans Rademacher

How to draw a tree (*Fibonacci series*), by Mr. W. J. Turanski

Farey series and lattice points, by Mr. Morris Newman.

The officers for the coming year are: President, William J. Turanski; Secretary, Mary-Elizabeth Hamstrom; Treasurer, Anthony Penico.

Pi Mu Epsilon, Syracuse University

The Pi Mu Epsilon Chapter at Syracuse University started the year with the initiation banquet at which thirty-eight students and faculty members were initiated. The speaker for the evening was Dr. S. S. Cairns, head of the mathematics department of the University, who gave a talk on *Peculiarities of polyhedra*.

Other papers given during the year included:

Irrational numbers, by Dr. Bruns

Operational calculus, by Dr. L. L. Lindsey, retiring head of the Applied Science Mathematics Department.

The group experimented this year by giving a Pi Mu Epsilon dance on April 25, 1947 for members and their guests. The decorations were illustrated mathematical problems hung on the walls, and each member wore a name tag. At the last meeting of the year, twenty-eight students were initiated.

The following officers were elected for the year 1947–48: Director, Joyce E. Jones; Vice-Director, Richard Stitt; Secretary, Elizabeth Gillespie; Treasurer, Donald Schick.

Officers for the group for the year 1946–47 were: Director, June Kath; Vice-Director, Sara D. Gray; Secretary, Joyce E. Jones; Treasurer, Jane Berkman.

Mathematics Club, Syracuse University

Papers presented by the members of the faculty were:

Quality control, by Dr. Hicks

Theory of complex numbers, by Dr. Bruns

Unsolved problems, by Dr. Rosenbloom

Four-color problem, by Dr. A. Gelbart

The mathematics of Mickey Mouse, by Dr. S. S. Cairns.

The annual Christmas party in the first term, and a hot-dog roast in the second comprised the outstanding social events of the year. At another meeting movies of various mathematical operations were shown.

Officers who served during the year are: President, Lorraine Zerveck; Vice-President, Sara Gray; Secretary, Teresa Hastings; Treasurer, Milton Fisher; Publicity Head, Elizabeth Gillespie.

Officers elected for the year 1947–48 are: President, Teresa Hastings; Vice-President, Elizabeth Gillespie; Secretary, Wallace Roher; Treasurer, John Dolphin; Publicity and Social Chairmen, Annalyse Haas and David Ashkar.

Mathematics Club, Oshkosh State Teachers College

The *Mathematics Club* of the Oshkosh State Teachers College resumed monthly meetings this year after a period of suspended activities. Members of the club presented the following papers:

Applications of mathematics to meteorology, by Robert Wonders

Use of Mathematics in the construction of a radio, by Alan Brunka

History of the normal probability curve, by Elsie Gandt

History of logarithms and the slide rule, by Clarence Bittner

Early history of mathematics, by Roy Matzdorf

The life and work of Einstein, by Kermet Jones.

The club made a study of motion pictures dealing with mathematics and the following films were shown:

Frequency curves

Gallup poll

Geometry brought to life

Locus

Rectilinear coördinates

Slide Rule

Origin of mathematics

Einstein's theory of relativity

One meeting was devoted to mathematical recreations and the year's activities closed with a picnic.

Officers for 1946–47 were: President, James W. Hartford; Vice-president, Harvey Stangel; Secretary, Louise M. Richter; Treasurer, Donald W. Derber; Faculty Adviser, Professor May M. Beenken.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

REPORT OF CONFERENCE OF THE AMERICAN COUNCIL ON EDUCATION

The annual Conference of Constituent Member Representatives of the American Council on Education was held in Washington, D. C., January 23–24, 1948. At this conference Professor A. A. Bennett of Brown University represented the Mathematical Association of America. A summary of Professor Bennett's report of the conference follows.

The first session was devoted to discussing a report on a questionnaire on Universal Military Training, issued by the American Council on Education and dealing with the recommendations of the President's Advisory Commission on Universal Training. The following essentials for national security were recommended: 1) a strong, healthy, educated population, 2) need for scientific research and development, 3) a coördinated intelligence service, 4) a regular Army, Navy, Air Force, and Marine Corps of high mobility and striking power, 5) industrial mobilization and stock piling. Less than a quarter of nearly every group polled were in favor of the proposed plan for universal military training.

Other sessions were devoted to the discussion of the following topics: 1) the extension of the social security act to apply to non-taxed educational organizations and systems; 2) discussion of "The Report of the President's Commission on Higher Education." This "Report" was urgently recommended for detailed study by all the constituent organizations.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1948:

Catholic University of America. June 28 to August 6: Professor Finan, theory of numbers; Dr. McBrien, theory of equations; Professor Ramler, synthetic projective geometry; Professor Rice, advanced calculus, calculus of observations; Mr. Slud, partial differential equations, calculus of variations.

Columbia University. July 6 to August 13: Professor Kasner, survey of modern mathematics, differential geometry; Professor Koopman, partial differential equations of mathematical physics; Dr. Levi, introduction to algebra; Professor Lorch, introduction to mathematical logic; Professor Murray, theory of functions of a complex variable; Professor Ritt, differential equations; Professor Wolfowitz, probability, statistical inference.

Iowa State College. June 14 to July 21: theory of equations; elementary differential equations I; analytical mechanics I; non-euclidean geometry; history of mathematics; advanced calculus I; vector analysis; functions of a real variable; advanced probability. July 21 to August 27: differential equations II;

analytical mechanics II; advanced calculus II; Laplace transform and operational mathematics, potential theory.

Northwestern University. The following advanced courses are offered during the six-week session, June 25 to August 7, and during the nine-week session, June 25 to August 28: fundamental concepts of analysis; determinants and matrices; non-euclidean geometry; theory of statistics; econometrics; introduction to the theory of numbers; algebra for teachers; the history and teaching of mathematics; differential equations of mathematical physics; introduction to modern algebra; vector analysis; independent study; seminar; thesis.

Ohio State University. June 21 to September 3: Professor Alden, theory of equations; Professor Bamforth, projective geometry, theory of matrices; Professor Mickle, differential geometry, continuous groups; Professors Miller and Alden, advanced calculus; Professors Sealander and Alden, differential equations; Professor Sealander, vector analysis.

Stanford University. June 17 to August 28: Professor Polya, elementary mathematics from higher point of view, selected topics from the theory of functions of a complex variable; Professor Schoenberg, advanced calculus, partial differential equations of physics and engineering; Professor Spencer, advanced calculus, elementary tensor calculus; Staff, seminar, advanced reading and research.

State University of Iowa. June 8 to August 4: Professor Chittenden, differential equations; Professor Conkwright, elementary group theory; Professor Cosby, Jr., calculus of variations; Professor Craig, matrices and determinants, sequential analysis; Professor Knowler, mathematics of business and industry; Professor Price, teaching of mathematics; Professor Woods, pure geometry; Professor Wylie, astronomy, reading in astronomy; Staff, research.

Teachers College, Columbia University. July 6 to August 13; Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, professionalized subject matter in advanced secondary school mathematics, teaching algebra in secondary schools; Dr. Lazar, history of mathematics, logic for teachers of mathematics; Mr. Mirick, elementary mechanics, teaching geometry in secondary schools; Professor Reeve, teaching and supervision of mathematics-junior high school, teaching and supervision of mathematics-senior high school. In addition, on consecutive Thursdays beginning on July 8, there will be five special lectures and discussions pertaining to the reorganization and teaching of mathematics in the post-war world.

University of Buffalo. June 14 to September 4: Professor Pound, analytic mechanics, functions of a complex variable. July 6 to August 14: Professor Gehman, functions of a real variable; Professor Schneckenburger, non-euclidean geometry.

University of Chicago. June 29 to September 4: Mr. Crabtree, integration in abstract spaces; Professor Kaplansky, introduction to modern higher algebra; Professor MacLane, introduction to algebraic topology; Professor Santaló, integral geometry; Professor Schilling, class-field theory; Professor Segal,

mathematical foundations of quantum theory; Professor Stone, theory of functions of a complex variable; Professor Weil, introduction to algebraic geometry, seminar on current literature.

University of Illinois. Professor Bourgin, algebra; Professor Brahana, theory of groups; Professor Chanler, geometry; Professor Day, analysis in function spaces, analysis; Miss Hildebrandt, the teaching of general mathematics in high school; Professor Mendel, introduction to higher geometry; Professor Munroe, introduction to higher analysis; Professor Schubert, introduction to higher algebra; Professor Vaughan, fundamental concepts.

University of Michigan. June 21 to August 13: The following advanced courses will be offered in addition to the standard courses in differential equations, theory of equations, advanced calculus, mechanics and statistics: Professor Bartels, Fourier series and vector analysis; Professor Brauer, higher algebra and seminar in algebra; Professor Copeland, probability; Professor Craig, analytic theory of frequency functions; Professor Dwyer, analysis of variance and computational methods; Professor Fischer, intermediate mathematics of life insurance; Professor Hay, advanced mechanics; Professor Hildebrandt, functions of a complex variable and orthogonal functions; Dr. Jones, teaching of algebra, history of geometry and teaching of elementary collegiate mathematics; Professor Kaplan, elementary function theory with applications; Dr. Lockhart, synthetic projective geometry; Professor Myers, differential geometry; Professor Rainville, intermediate differential equations and number theory; Professor Thrall, theory of vector spaces; Professor Wilder, foundations of mathematics and topology of polyhedra and the n-sphere.

University of Minnesota. June 15 to July 23: Professor Carlson, advanced analytic geometry, solid analytic geometry; Professor Hatfield, mathematical recreations, mathematics of small vibrations; Professor Loud, differential equations, numerical methods in computation. July 26 to August 28: Professor Bearman, vector analysis, the calculus of finite differences; Professor Olmstead, intermediate calculus.

University of North Carolina. June 10 to July 20: Professor Brauer, functions of a complex variable; Professor Cameron, introduction to modern algebra; Professor Garner, history of mathematics; Professor Henderson, elements of non-euclidean geometry; Professor Hill, elementary mathematical statistics; Professor Hoyle, differential equations, and advanced calculus I; Professor Winsor, college geometry. From July 21 to August 28: Professor Browne, algebraic invariants; Professor Hobbs, theory of equations; Professor Lasley, differential geometry; Professor Linker, differential equations (continued); Professor Mackie, advanced calculus I (continued).

University of Pennsylvania. Professor Beal, modern analytic geometry; Professor Caris, theory of numbers; Professor Cohen, differential equations, theory of functions of a complex variable.

University of Virginia. June 28 to August 21: Mr. Floyd, differential equations and applied mathematics; Professor Hedlund, advanced analysis; Profes-

sor Whyburn, analytic topology.

University of Wisconsin. June 25 to August 20: Professor Arnold, mathematics of elementary statistics; Professor Bruck, college geometry, determinants and matrices; Professor Langer, Lie theory of differential equations; Professor MacDuffee, survey of the foundations of arithmetic, tensor analysis; Mrs. Sokolnikoff and Professor Specht, higher mathematics for engineers; Staff, mathematical applications.

University of Wyoming. June 14 to July 16: Professor Barr, methods in secondary mathematics; Professor Bristow, advanced calculus; Professor Schwid, ordinary differential equations; Professor Smith, vector analysis. July 19 to August 20: Professor Neubauer, history of mathematics; Professor Varineau, partial differential equations, curve fittings.

REPRINTS AVAILABLE

Reprints of the following articles may be secured by writing to the office of the Secretary-Treasurer, University of Buffalo, Buffalo 14, New York. They will be distributed free to members and subscribers as long as the supply lasts.

"A List of Mathematical Books for Schools and Colleges" (1917);

"A Suggested List of Mathematical Books for Junior College Libraries" (1925);

"Benjamin Peirce, 1809–1880." By R. C. Archibald (1925);

"Report of the Committee on Assigned Collateral Reading in Mathematics" (1928);

"Report of the Committee on College Entrance Requirements in Geometry" (1931);

"Collegiate Mathematics Needed in the Social Sciences" (1932);

"Readings in the Literature on Teaching with Special References to Mathematics" (1935);

"Advanced Preparatory Mathematics in England, France and Italy" by W. D. Cairns (1935);

"The Ph.D. Degree and Mathematical Research," by R. G. D. Richardson (1936);

"Herbert Ellsworth Slaught," by W. D. Cairns and G. A. Bliss (1938);

"It Can't Happen Here," a mathematical musical farce, by A. Marie Whelan (1938);

"Report of the Committee on Tests" (1940).

PERSONAL ITEMS

Professor R. W. Brink represented the Association at the dedication of a new Science Hall on March 8–9, 1948 at the College of St. Thomas, St. Paul, Minnesota.

Professor P. A. Caris was appointed a delegate of the Association to the meeting of the American Academy of Political and Social Science at Philadelphia on April 2–3, 1948.

Professor J. W. Cell represented the Association at the inauguration of

John D. Messick as president of the East Carolina Teachers College, Greenville, North Carolina on March 6, 1948.

Professor B. P. Reinsch was a representative of the Association at the inauguration of J. Hillis Miller as president of the University of Florida, Gainesville, Florida on March 5, 1948.

Dr. John H. Curtiss, chief of the National Applied Mathematics Laboratories of the National Bureau of Standards, was elected a fellow of the American Statistical Association at the Association's annual meeting, December 28–31, 1947.

Dr. Mina Rees and Dr. Warren Weaver were included on the list of American citizens receiving British awards "in recognition of valuable services rendered to the Allied war effort in various fields of scientific research and development." The list was announced by Lord Inverchapel, the British Ambassador, on February 11, 1948. Both Dr. Rees and Dr. Weaver were awarded the King's Medal for Service in the Cause of Freedom.

Stanford University announces the following: Professor Harold Davenport from University College, London is at Stanford University for the year 1947–48 as Visiting Professor; Professor Ainsley H. Diamond of Oklahoma Agricultural and Mechanical College is teaching part-time at Stanford University during the spring quarter, 1947–48; Professor Rhoda Manning is on leave from Oregon State College and is teaching part-time during the year 1947–48 at Stanford University; Professor M. Schiffer of Harvard University is giving a series of lectures from February 1 to June 15, 1948. Other lectures have been given recently at Stanford University by Professor Harold Cramer of the University of Stockholm, Sweden; Professor M. Fréchet, Institut Henri Poincaré, Paris; Professor Marcel Riesz of the University of Lund, Sweden.

Professor Richard Brauer of the University of Toronto has been appointed to a professorship at the University of Michigan.

Dr. R. S. Burington is Chief Mathematician and Director of the Evaluation and Analysis Group, Bureau of Ordnance, Navy Department, Washington, D. C.

Associate Professor T. F. Cope of Queens College has been promoted to a professorship.

Professor H. A. Davis of West Virginia University has been appointed Head of the Department of Mathematics.

Dr. Henri A. Jordan has been appointed to an associate professorship at Colby College.

Associate Professor H. E. Jordan was reported erroneously to have left the University of Kansas.

Professor G. E. Raynor of Lehigh University will become Head of the Department of Mathematics on July 1, 1948.

Professor J. B. Reynolds, head of the Department of Mathematics of Lehigh University, will retire on June 30, 1948.

Mr. Emil D. Schell has accepted an appointment as chief of the Mathe-

matics and Electronic Computor Branch, Office of the Comptroller, United States Air Forces.

Assistant Professor L. W. Swanson of Coe College has been promoted to an associate professorship.

Professor John L. Synge has been appointed to a Senior Professorship in the School of Theoretical Physics at the Dublin Institute for Advanced Studies. He is resigning from his present position as Head of the Department of Mathematics at Carnegie Institute of Technology as of the end of the academic year, and will take up his duties in Dublin on September 1.

Assistant Professor Leo Zippin of Queens College has been promoted to an associate professorship.

The University of Michigan announces the appointment of S. W. Hahn as instructor.

Mr. A. L. McCarty, formerly of City College of San Francisco, died on June 20, 1947. Mr. McCarty, a charter member of the Association, was one of the organizers of the Northern California section.

Professor Emeritus A. B. Turner of the College of the City of New York died on February 5, 1948 at the age of seventy-five years. He was a charter member of the Association.

Professor G. E. Wahlin of the University of Missouri died on February 11, 1948. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE INDIANA SECTION

The fall joint meeting of the Indiana Section of the Mathematical Association of America and the Mathematics Section of the Indiana Academy of Science was held at Ball State Teachers College, Muncie, Indiana on October 17, 1947.

Sixty-four persons attended the meeting, including the following twenty-one members of the Association: W. L. Ayres, Juna L. Beal, G. E. Carscallen, K. W. Crain, Olive M. Draper, P. D. Edwards, E. L. Godfrey, G. H. Graves, H. F. S. Jonah, M. W. Keller, F. C. Leone, P. M. Pepper, J. C. Polley, D. H. Porter, C. K. Robbins, A. E. Ross, L. S. Shively, W. O. Shriner, Florence A. Wirsching.

Announcement of the 1948 national meeting of the National Council of Teachers of Mathematics to be held in Indianapolis on April 2–3, 1948, was made by M. H. Ahrendt, of Anderson College.

No business meeting was held. The time and place of the spring meeting for 1948 was to be arranged by the officers of the Section.

The following papers were presented.

1. *Pythagorean triangles*, by John Funderburg and G. A. Jackson, introduced by the Secretary.

In the euclidean plane, an assemblage of points representing Pythagorean triangles is shown to consist of a parabolic lattice set and sets that are multiples of this set. A subset contains all primitive triangles. It is demonstrated that this assemblage of points can be located on systems of hyperbolas, circles, and straight lines. The approximate values of certain constants (including roots of prime numbers, transcendental e and π) are shown to be obtained by the use of convergent or oscillatory series, of which the members are represented by points included in this assemblage of points.

2. *Some special sums of cotangents*, by H. F. S. Jonah, Purdue University.

The speaker illustrated a method for summing certain finite sums of cotangents. The original sums arose in a research project in electrical engineering.

3. *Mathematics teacher training in relation to the proper teaching of undergraduate algebra*, by A. E. Ross, University of Notre Dame.

The speaker discussed a fundamental approach to the teaching of algebra in college, as well as the problems of training teachers to carry out such a program of instruction successfully.

4. *A study of factors related to engineering mathematics at Purdue University*, by Paul Irick, Purdue University, introduced by the Secretary.

The speaker showed the relation between grades in mathematics and various factors such as position of student in graduating class, average grade in high school mathematics, number of high school units in mathematics, and grades on different tests given during orientation period. The study followed the students through the first two years of college mathematics.

5. *A mathematical theory of religion*, by G. H. Graves, Purdue University.

Due to the studies of Whitehead, Russell, Keyser, and others, it is now generally recognized that mathematics has no particular subject matter but is concerned with constructing logical systems on postulates suggested by any field of interest. Religion is a promising field in this connection, for in religion, we constantly observe conclusions and decisions, and hence conduct and character, resulting from postulates held as convictions by an individual or by a society. Just as geometry, for instance, has gained greatly in clearness and in range by a study of its foundations and the recognition of incompatible systems which are nevertheless consistent individually, so it may be expected that different religions can gain in clearness and in tolerance by studying their fundamental postulates with a view to eliminating contradictions and non-essentials, and to tracing the connections of the characteristics of individuals or societies with their fundamental postulates.

6. *A reduced set of postulates for hyperbolic geometry*, by Rev. H. F. DeBaggis, C.S.C., University of Notre Dame, introduced by the Secretary.

The speaker presented a minimal set of postulates for hyperbolic geometry. Independence examples were given for all of the postulates.

7. *The postulates of a tri-operational algebra*, by Rev. F. L. Brown, C.S.C. University of Notre Dame, introduced by the Secretary.

In this paper the speaker presented a set of postulates for a tri-operational algebra, showing the independence of each of the postulates. He presented the minimal set satisfying these postulates and derived a few elementary consequences. (These consequences are among those published by the author in the *Reports of a Mathematical Colloquium*, Issue 5–6, Issue 7, Notre Dame, Indiana.)

8. *Functional representation of partially ordered additive groups*, by Ky Fan, University of Notre Dame, introduced by the Secretary.

For any compact Hausdorff space Ω, the totality $C(\Omega)$ of all real continuous functions defined on Ω may be considered as a partially ordered additive group (p.o.a.g.). Any subgroup G of $C(\Omega)$ which contains all constant functions is obviously a p.o.a.g. with the following three properties: (1) G contains a sub-group R which is group-order-isomorphic to the totally ordered additive group of all real numbers; (2) The sub-group R contains an element e such that for any element f of G, the relation $ne \geqq f$ holds for some natural number n; (3) If for some pair of elements f, g of G, $nf+g \geqq 0$ holds for all natural numbers n, then $f \geqq 0$. Conversely, for any abstract p.o.a.g. G with properties (1), (2), (3), there exists a compact Hausdorff space Ω such that G is group-order-isomorphic to a sub-group G' of the p.o.a.g. $C(\Omega)$ formed by all real continuous functions on Ω, where G' contains all constant functions.

9. *Geometric illustrations of abstract complexes*, by Charles Brumfiel, Ball State Teachers College, introduced by the Secretary.

The speaker gave two and three dimensional examples of abstract topological complexes. The incidence matrices of an n-complex completely determine its topology. Methods were explained for calculating topological invariants, Betti numbers, and torsion coefficients, by means of the incidence matrices.

P. M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948
INDIANA
IOWA
KANSAS
KENTUCKY, Berea, May, 1948
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
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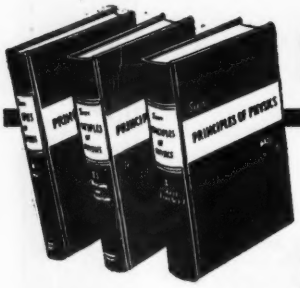

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